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Iterative learning control for non-linear systems with deadzone input and time delay in presence of measurement noise

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Abstract: The main purpose of this study is to propose an iterative learning control (ILC) algorithm for non-linear systems with dead-zone input and time delay in presence of measurement noise. The dead-zone non-linearity is described by a general model and all parameters of the dead-zone are unknown. The state function is allowed to grow as fast as any polynomial with arbitrary order, and thus does not satisfy the global Lipschitz condition. The time delay may be time varying, and multiple time delays are also considered. It is proved that the ILC algorithm given in the study converges to the optimal one minimising the tracking error in presence of measurement noise almost surely. An illustrative numerical example is presented to show the effectiveness.

1 Introduction

Iterative learning control (ILC) is an advanced control method, designed to handle those systems that could complete some tasks over a fixed time interval and repeatedly performed, for example the robot arms, chemical processes, hard-disk drivers, servo systems etc. For such kind of control systems, ILC may combine previous input and output information to adjust the current control input, so that the tracking performance could be improved as the number of cycles increases. It is much more attractive that the ILC not only requires less prior knowledge of control systems but it acts effectively. Since introduced by Arimoto *et al.* [1] in 1984, it has drawn much attention from researchers and has successfully been applied in practice, see [2–4] and references therein.

Among the numerous papers on ILC, there are few papers focus on time-delay systems (TDS). TDS, however, are common in many practical applications, which motivates research on ILC for TDS [5–14]. The convergence of ILC for continuous-time LTI system with an extra delayed state term in system equation is studied in [5], where state tracking problem is considered and a delayed tracking error term is embedded in control update law to cancel the influence of time-delayed states. A some strict condition on the two delay terms is required to ensure the convergence in the sense of l_2 norm. However, how to satisfy the condition in practice is not addressed clearly there. High-order ILC has been investigated for non-linear time-varying continuous systems in [6], and sufficient conditions are provided to guarantee that the tracking errors are bounded by reinitialisation errors, uncertainties and disturbances to systems. Similar systems have been discussed in [7, 8]. In [7], the arbitrary relative degree is considered and the ILC law is constructed with corresponding derivative orders whereas Sun and Wang [8] provide a D-type ILC update law for system with relative degree 1. Based on conventional contraction-type technique, it has been proved that the ILC would converge under critical assumptions, one of which is that all non-linear functions satisfying globally Lipschitz condition (GLC) [6–8]. Multiple time delays of linear continuous-time systems have been considered in [9–12] on the basis of 2D theory [15]. Necessary and sufficient conditions for convergence of the proposed ILC are given in [9] where systems are with state and with input delays, respectively. Meng *et al.* [10, 11] deal with the similar systems with initial shifts, where the input update law is D-type adding a pure error term or an initial rectifying term in [10] and is PD-type with known states information in [11]. All ILC convergence results of [9–11] are established on the basis of a convergence result for 2D linear continuous–discrete systems. In [12], the authors mainly focus on robust ILC algorithm for uncertain systems with time delay based on Lyapunov-like approach. Besides, it is also worth to point out that ILC has been discussed for systems with time delays and sufficient stability conditions are provided by LMI from frequency domain in [13, 14]. However, none of the above works discusses the stochastic measurement noise and the essential non-linear dependence of output on input.

Dead-zone is a class of important non-smooth non-linearity, which is commonly found in various engineer systems, especially in actuators, such as hydraulic and pneumatic servo valve, electric servo systems etc. The dead-zone may limit the system performance, therefore research on this topic is much needed. In [16], affine non-linear systems with dead-zone input are investigated, and ILC algorithm would converge when all non-linear functions satisfy GLC. Another work [17] proposes a new ILC algorithm for affine non-linear systems with dead-zone input but without GLC on non-linear functions. Meanwhile, there are some works to handle dead-zone input from the point of view of adaptive control [18–21].

In practical applications, various noises may enter the processes. For example, the output signals are often corrupted by stochastic measurement noise. Owing to randomness and unboundedness of statistical noise, most existing ILC algorithms are not appropriate for such case. This motivates research on stochastic ILC [22–25], all for discrete-time systems. The recursive algorithm for learning gain matrix is given in [22, 23] based on the derivative of the covariance matrix of the minimal (in the least-square sense) tracking error with respect to learning gain matrix for linear stochastic systems, which leads to the ILC algorithm, and the convergence in the mean-square sense is established. The stochastic system is also considered in [24, 25], which, requiring no information about system matrices, gives a new type of ILC algorithm based on the Kiefer–Wolfowitz algorithm from stochastic approximation and proves its convergence with probability one to the optimal control. It is worth to point out that Saab [23] and Chen and Fang [25] consider affine non-linear systems. However, the outputs of the systems considered in [23, 25] depend on the inputs, in essence, in a linear way. Furthermore, all the above research on stochastic ILC has focused on systems with no time delays.

The main purpose of this paper is to design an ILC algorithm for non-linear systems with dead-zone input non-linearity and state time delays in presence of measurement noise. Stochastic approximation is the basic tool for our convergence analysis, which is, in essence, of great help to remove the GLC requirements of the non-linear functions. Unlike [24, 25] the Robbins–Monro algorithm is adopted as it could help us to avoid the non-smooth non-linearity of dead-zone input function. The main contributions of the paper are as follows.

- The dead-zone non-linearity in our paper is described by a general non-linear dead-zone model, which may let it be hard to construct a dead-zone inverse. All parameters of the dead-zone are supposed to be unknown.
- Uncertain time delays in this papers may be time varying, which have not been handled in [5–11]. One possible reason may be that all the above papers consider continuous-time systems.
- The non-linear functions of the system are allowed to grow as fast as any polynomial with arbitrary order, wherefore GLC is no longer satisfied. Thus, the ILC algorithms designed in [5–12] cannot be applied here.
- Compared with previous stochastic ILC results [22–25], in this paper, the output depends on the input in a non-linear way because of dead-zone input, which also is a non-smooth non-linearity. Besides, time delays are also considered.

The rest of the paper is arranged as follows. The problem formulation together with general dead-zone model and optimal input are given in Section 2. The ILC algorithm is defined in Section 3, and its convergence analysis is also provided there. A numerical illustration is provided in Section 4. Finally, some concluding remarks are given in Section 5.

The following notational conventions will be used in this paper. $\mathbb{R}$ denotes the field of all real numbers and $\mathbb{R}^n$ denotes the field of all n-dimensional vectors. By $\|\cdot\|$ we denote the Euclidean norm. $\mathbb{E}$ denotes mathematical expectation operator. By $A_k \underset{k\to\infty}{\longrightarrow} A$ we mean that the sequence $\{A_k\}$ converges to A as $k \to \infty$. In the paper all convergence results are in the sense of almost surely (a.s.) unless specified otherwise.

2 Problem formulation and preliminaries

In this section, we will first formulate single-input-single-output (SISO) delayed affine non-linear system with dead-zone input, followed by suitable assumptions, and then define the optimal control sequence under tracking performance index. Note that only the outputs are available for ILC algorithms, in other words, no state measurements are required.

2.1 Model and dead-zone input

Consider a class of SISO affine non-linear time-varying TDS with dead-zone input in the following form

$$\begin{cases} x_k(t+1) = f(t, x_k(t-\tau_t)) + b(t, x_k(t-\varsigma_t))v_k(t) \\ v_k(t) = \mathcal{D}(u_k(t)) \\ y_k(t) = c(t)x_k(t) + w_k(t) \end{cases} \tag{1}$$

where $\mathcal{D}(\cdot)$ denotes dead-zone function defined later. Subscripts $k = 0, 1, 2, \ldots$ denote the different cycles, whereas $t \in [0, T]$ denotes an arbitrary time instance in a cycle. $u_k(t) \in \mathbb{R}$ and $y_k(t) \in \mathbb{R}$ are the input and output, respectively, whereas $x_k(t) \in \mathbb{R}^n$ is the state vector. $w_k(t)$ is measurement noise and $v_k(t)$ is unknown intermediate signal. τ_t and ς_t are unknown time-varying delays. $f(,\ )$: $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a system non-linear function; $\boldsymbol{b}(\cdot,\cdot)$ and $\boldsymbol{c}(\cdot)$ are column vector and row vector, respectively.

The dead-zone function $\mathcal{D}(\cdot)$ with the input u is shown in Fig. 1 and defined as follows

$$v = \mathcal{D}(u) = \begin{cases} g_r(u), & u \in I_r \triangleq [b_r, \infty) \\ 0, & u \in I_m \triangleq [b_l, b_r] \\ g_l(u), & u \in I_l \triangleq (-\infty, b_l] \end{cases} \tag{2}$$

For the sake of simplicity, we need the following assumption on dead-zone function.

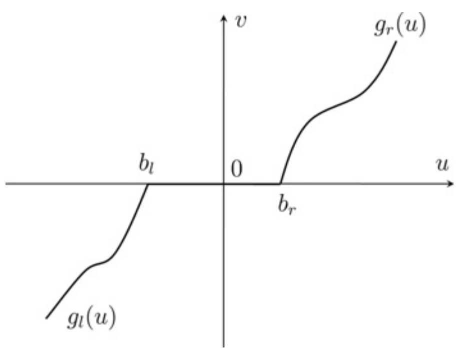

Fig. 1 *Non-symmetric non-linear dead-zone*

A1: The dead-zone parameters b_l and b_r are unknown constants, but their signs are known, that is $b_r > 0$ and $b_l < 0$. The growth of the dead-zone's left and right functions, $g_l(\cdot)$ and $g_r(\cdot)$, is continuous and strictly increasing.

Remark 1: Note that such kind of dead-zone has also been discussed by Zhang and Ge [26, 27], but the differentiability of $g_l(\cdot)$ and $g_r(\cdot)$ in [26, 27] is not required here.

Let $\{\mathcal{F}_k\}$ be the non-decreasing σ-algebras defined by

$$\mathcal{F}_k \triangleq \sigma(y_i(t), x_i(t), w_i(t), \quad 0 \le i \le k, t \in [0, T])$$

Define the set of admissible controls as follows

$$U = \{u_{k+1}(t) \in \mathcal{F}_k, \sup_k |u_k(t)| < \infty \text{ a.s.}, \\ t \in [0, T-1], k = 0, 1, 2, \ldots\} \quad (3)$$

The control purpose is to find $\{u_k(t), k = 0, 1, 2, \ldots\} \in U$ such that the following tracking performance index is minimised

$$V_t(u_k(t)) = \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |y_k(t) - y_d(t)|^2 \quad \text{a.s.} \quad (4)$$

where $y_d(t)$, $t \in [0, T]$, is the target signal to track.

To propose the ILC, the following assumptions are to be used.

A2: Tracking target $y_d(t)$ is realisable, that is for an appropriate initial value $x_d(t)$, there exists control input $\{u_d(t)\}$ generating the trajectory for the nominal plant. That is, the following difference equations are satisfied

$$\begin{cases} x_d(t+1) = f(t, x_d(t-\tau_t)) + \boldsymbol{b}(t, x_d(t-\varsigma_t))v_d(t) \\ v_d(t) = \mathcal{D}(u_d(t)) \\ y_d(t) = \boldsymbol{c}(t)x_d(t) \end{cases} \quad (5)$$

A3: The value of $\boldsymbol{c}(t+1)\boldsymbol{b}(t, x(t-\varsigma_t))$ is unknown, but its sign is known and non-vanishing. Without loss of generality, assume $\boldsymbol{c}(t+1)\boldsymbol{b}(t, x(t-\varsigma_t)) > 0$.

A4: Non-linear functions $f(\cdot,\cdot)$ and $\boldsymbol{b}(\cdot,\cdot)$ satisfy the following conditions with the second variable

$$\|f(t, x) - f(t, y)\| \le \sum_{i=1}^{l} m_i \|x - y\|^i \quad (6)$$

$$\|\boldsymbol{b}(t, x) - \boldsymbol{b}(t, y)\| \le \sum_{i=1}^{l} n_i \|x - y\|^i \quad (7)$$

where l is some unknown positive integer, whereas m_i and n_i, $i = 1, 2, \ldots, l$ are unknown positive constants.

Note that the GLC can be thought as a special case of A4 with $l = 1$. However, the GLC is no longer satisfied when A4 holds, thus the results in [6–8] cannot be applied here.

A5: For any $t \in [0, T]$, the measurement noises $\{w_k(t), t \in [0, T], k = 1, 2, \ldots\}$ are independent along the cycle index with zero mean and finite second moments

$$\mathbb{E}w_k(t) = 0, \quad \forall t \in [0, T], \forall k = 1, 2, \ldots, \\ \mathbb{E}w_k^2(t) = R_t, \quad \forall k = 1, 2, \ldots \quad (8)$$

Note that assumption A5 is suitable since the properties of the noises are mainly focused on different cycles of fixed time rather than different times of fixed cycle. Thus the measurement noises may depend on time. It is obvious that zero-mean white noise satisfies the assumption.

A6: Initial values are re-initialised asymptotically, that is $x_k(0) - x_d(0) \underset{k\to\infty}{\longrightarrow} 0$.

A7: The unknown state time-varying delays τ_t and ς_t satisfy

$$0 \le \tau_t \le t, \quad 0 \le \varsigma_t \le t \quad (9)$$

From A7 we know that the current state of time t only depends on some state of time t' with $0 \le t' \le t-1$. In other words, the current state of time t only depends on the states that generated by the state equation in one cycle. Since nothing is known about the states before time 0, assumption A7 is suitable and makes the following part of the paper more readable. Roughly speaking, A7 could be removed if we assume that $x_k(-m) = x_d(-m)$, $\forall m > 0$.

For expression convenience, denote

$$f_k(t) \triangleq f(t, x_k(t-\tau_t)), \quad f_d(t) \triangleq f(t, x_d(t-\tau_t))$$

$$\boldsymbol{b}_k(t) \triangleq \boldsymbol{b}(t, x_k(t-\varsigma_t)), \quad \boldsymbol{b}_d(t) \triangleq \boldsymbol{b}(t, x_d(t-\varsigma_t))$$

$$\delta x_k(t) \triangleq x_d(t) - x_k(t), \quad \delta v_k(t) \triangleq v_d(t) - v_k(t), \\ \delta u_k(t) \triangleq u_d(t) - u_k(t)$$

$$e_k(t) \triangleq y_d(t) - y_k(t), \quad \delta f_k(t) \triangleq f_d(t) - f_k(t), \\ \delta \boldsymbol{b}_k(t) \triangleq \boldsymbol{b}_d(t) - \boldsymbol{b}_k(t)$$

$$\boldsymbol{c}^+\boldsymbol{b}_k(t) \triangleq \boldsymbol{c}(t+1)\boldsymbol{b}(t, x_k(t-\varsigma_t)), \\ \boldsymbol{c}^+f_k(t) \triangleq \boldsymbol{c}(t+1)f(t, x_k(t-\tau_t)$$

2.2 Optimal control sequence

In this subsection, the minimum of the tracking performance index (4) is proposed, and then we will point out how to achieve this minimum.

We first characterise the uniqueness of optimal intermediate signal $v_d(t)$.

By A2, A3 and A7, it is seen that the following intermediate signal is uniquely defined, although it is not directly available

$$v_d(i) = (\boldsymbol{c}(i+1)\boldsymbol{b}(i, x(i-\varsigma_t)))^{-1}(y_d(i+1) \\ - \boldsymbol{c}(i+1)f(i, x_d(i-\tau_t)))$$

with the initial value

$$v_d(0) = (\boldsymbol{c}(1)\boldsymbol{b}(0, x(0)))^{-1}(y_d(1) - \boldsymbol{c}(i+1)f(i, x_d(0)))$$

Recall (2) and A1–A2 we find, however, that there maybe existing more than one control input $u_d(i)$ such that $v_d(i) = \mathcal{D}(u_d(i))$ when $v_d(i) = 0$. However this does not affect the development of the subsequent derivations of this paper (see Theorem 1 below for some details).

The following two lemmas are needed before we give the condition which the optimal control sequence satisfies.

Lemma 1 (Theorem 2.8 in [28]): Let $\{X(t), \mathcal{F}_t\}$ be a martingale difference sequence (MDS) and $\{M(t), \mathcal{F}_t\}$ an adapted sequence, $\|M(t)\| < \infty$, $\forall t \ge 0$. If $\sup_{t\ge0} \mathbb{E}[\| X(t)\|^2$

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$|\mathcal{F}_{t-1}] < \infty$, a.s., then as $n \to \infty$

$$\sum_{t=0}^{n} M(t)X(t+1) = O\left(\left(\sum_{t=0}^{n} \|M(t)\|^2\right)^{(1/2)+\eta}\right) \quad \text{a.s.}$$

where η is an arbitrary positive real number. For the concepts of MDS and adapted sequence the readers may refer to [29, 30].

Lemma 2: Consider system (1) and assume A2–A7 hold. If $\lim_{k\to\infty}|\delta v_k(s)| = 0$, $s = 0, 1, \ldots, t$, then at time $t+1$

$$\|\delta x_k(t+1)\| \underset{k\to\infty}{\longrightarrow} 0, \qquad \|\delta f_k(t+1)\| \underset{k\to\infty}{\longrightarrow} 0,$$
$$\|\delta \boldsymbol{b}_k(t+1)\| \underset{k\to\infty}{\longrightarrow} 0$$

Proof: We now inductively prove the conclusion. By (1) and (5)

$$\begin{aligned}\delta x_k(t+1) &= f_d(t) - f_k(t) + \boldsymbol{b}_d(t)v_d(t) - \boldsymbol{b}_k(t)v_k(t)\\ &= \delta f_k(t) + \delta\boldsymbol{b}_k(t)v_d(t) + \boldsymbol{b}_k(t)\delta v_k(t)\end{aligned} \tag{10}$$

From A7, we know that τ_0 and ς_0 equal 0. Thus for $t = 0$, noticing A4 and A6, we have

$$\|f_d(0) - f_k(0)\| \le \sum_{i=1}^{l} m_i\, \|x_d(0) - x_k(0)\|^i \underset{k\to\infty}{\longrightarrow} 0$$

$$\|\boldsymbol{b}_d(0) - \boldsymbol{b}_k(0)\| \le \sum_{i=1}^{l} n_i\, \|x_d(0) - x_k(0)\|^i \underset{k\to\infty}{\longrightarrow} 0$$

which implies the first and second terms on the right side of (10) tend to zero asymptotically. Since

$$\|\boldsymbol{b}_k(0)\| \le \|\boldsymbol{b}_d(0)\| + \|\delta\boldsymbol{b}_k(0)\|$$

it follows that $\boldsymbol{b}_k(0)$ is bounded. Incorporating with $\lim_{k\to\infty} |\delta v_k(0)| = 0$, the third term on the right side of (10) tends to zero. Hence, $\|\delta x_k(1)\| \underset{k\to\infty}{\longrightarrow} 0$ and by A4, $\|\delta f_k(1)\| \underset{k\to\infty}{\longrightarrow} 0$, $\|\delta\boldsymbol{b}_k(1)\| \underset{k\to\infty}{\longrightarrow} 0$. That is, the conclusions are valid for $t = 0$.

Assume that the conclusions hold for $s = 0, 1, \ldots, t-1$. Now we will show that the conclusions are true for t. By the induction assumption, $\|\delta x_k(s)\| \underset{k\to\infty}{\longrightarrow} 0$, $\|\delta f_k(s)\| \underset{k\to\infty}{\longrightarrow} 0$, $\|\delta b_k(s)\| \underset{k\to\infty}{\longrightarrow} 0$, $s = 0, 1, \ldots, t$. By the same argument as that used above, it can be shown that $\|\delta x_k(t+1)\| \underset{k\to\infty}{\longrightarrow} 0$, and therefore $\|\delta f_k(t+1)\| \underset{k\to\infty}{\longrightarrow} 0$, $\|\delta\boldsymbol{b}_k(t+1)\| \underset{k\to\infty}{\longrightarrow} 0$. This completes the proof. □

Theorem 1: Consider system (1) and tracking performance index (4) and assume A1–A7 hold. Then for any control sequence $\{u_k(t)\}$

$$V_t(\{u_k(t)\}) \ge R_t \quad \text{a.s. } \forall t$$

Furthermore, if the control sequence $\{u_k^0(t)\}$ satisfies

$$\begin{aligned}\delta v_k^0(t) &\triangleq v_d(t) - v_k^0(t) \underset{k\to\infty}{\longrightarrow} 0,\\ v_k^0(t) &= \mathcal{D}(u_k^0(t)), \quad t = 0, 1, \ldots, T-1\end{aligned} \tag{11}$$

then $\{u_k^0(t)\}$ is optimal, that is

$$V_t(\{u_k^0(t)\}) = R_t \quad \text{a.s. } \forall t$$

Proof: By A5 and the definition of $\mathcal{F}_k$, it follows that $\mathcal{F}_k$ is independent of $\{w_l(t), l = k+i, i = 1, 2, \ldots, \forall t \in [0, N]\}$, thus $\{w_k(t), \mathcal{F}_k\}$ is a MDS with $\sup_k \mathbb{E}[|w_k(t)|^2|\mathcal{F}_{k-1}] < \infty$ a.s. Meanwhile, inputs, outputs and state vectors are all adapted to $\mathcal{F}_k$. Therefore by (1)

$$\begin{aligned}&\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|y_k(t)-y_d(t)|^2\\ &=\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|\boldsymbol{c}(t)(x_k(t)-x_d(t))-w_k(t)|^2\\ &=\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|\boldsymbol{c}(t)\delta x_k(t)|^2(1+o(1))+\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|w_k(t)|^2\\ &\ge\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|w_k(t)|^2=R_t\end{aligned}$$

where the second equation holds by Lemma 1, and $o(1) \underset{n\to\infty}{\longrightarrow} 0$. The sufficient and necessary condition to achieve the minimum is

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}|c(t)\delta x_k(t)|^2 = 0$$

which is true when $\|\delta x_k(t)\| \underset{k\to\infty}{\longrightarrow} 0$. While the latter holds, if the control sequence $\{u_k^0(t)\}$ satisfies (11) by Lemma 2. □

3 ILC and its convergence

We first propose the ILC algorithm. Let M_k be positive real numbers such that $M_{k+1} > M_k$ and $M_k \underset{k\to\infty}{\longrightarrow} \infty$. Define the ILC algorithm as follows

$$u_{k+1}(t) = [u_k(t) + a_k e_k(t+1)]I_{[|u_k(t)+a_k e_k(t+1)|\le M_{\sigma_k(t)}]} \tag{12}$$

$$\sigma_k(t) = \sum_{i=1}^{k-1} I_{[|u_i(t)+a_i e_i(t+1)|>M_{\sigma_i(t)}]}, \quad \sigma_0(t) = 0 \tag{13}$$

where $a_k = 1/k$ is the learning gain. I_A is the indicator of random event A, defined by

$$I_A = \begin{cases}1, & \text{if random event } A \text{ holds}\\ 0, & \text{otherwise}\end{cases}$$

Here, by definition, $e_k(t) = y_d(t) - y_k(t)$.

Remark 2: The above-mentioned algorithm is a stochastic approximation algorithm with expanding truncations (SAAWET) [29]. a_k is defined here both as iterative learning gain and noise effect canceller. If the sign of

$c^+ b_k(t)$ is known to be negative, one can just replace the term $u_k(t) + a_k e_k(t+1)$ with $u_k(t) - a_k e_k(t+1)$.

For arbitrary fixed t, (12) can be rewritten as follows

$$u_{k+1}(t) = [u_k(t) + a_k c^+ b_k(t)(v_d(t) - v_k(t)) - a_k w_k(t+1) + a_k \varphi_k(t)] \times I_{[|u_k(t)+a_k c^+ b_k(t)(v_d(t)-v_k(t))-a_k w_k(t+1)+a_k\varphi_k(t)| \le M_{\sigma_k(t)}]} \tag{14}$$

where

$$\varphi_k(t) = c^+ \delta f_k(t) + c^+ \delta b_k(t) v_d(t)$$

Notice that $c^+ b_k(t) > 0$ always holds by A3. The noise includes two parts, one of which is $w_k(t+1)$ called measurement noise and the other is $\varphi_k(t)$ called system noise. Define the regression function as follows

$$L_{t,k}(u) \triangleq c^+ b_k(t)(v_d(t) - v) = c^+ b_k(t)(\mathcal{D}(u_d(t)) - \mathcal{D}(u)) \tag{15}$$

It is worth to point out that the regression functions depend on cycle index k for any fixed t, but their sets of roots do not.

Remark 3: Now we give a brief interpretation why our approach is effective for general non-linearity. Contraction mapping method (CMM) is applied by many previous works to guarantee the convergence of the control sequence, which in essence is obtaining a contraction-type inequality of $\|\delta u_k(t)\|$. Hence GLC is a technical requirement (see [6–8] for detailed derivations). However our approach is based on stochastic approximation which aims to find the root of an unknown function recursively. Thus GLC is no longer needed since the regression function (15) does not depend on $\|\delta f_k(t)\|$. In addition, the learning gain a_k in (12) corresponding to the contraction coefficient of CMM would guarantee the convergence.

To analyse the convergence of (14), Theorem 2.2.4 from [29] will be used. For reading convenience, we quote it and rewrite it as Lemma 3.

Lemma 3: For any fixed $t \in [0, T]$, assume that (13) and (14) satisfy the following conditions:

1. There is a continuously differentiable Lyapunov function (not necessarily being non-negative) $\zeta(\cdot)$: $\mathbb{R} \to \mathbb{R}$ such that

$$\sup_k \sup_{\delta \le d(u, J_t) \le \Delta} \zeta_u(u) L_{t,k}(u) < 0 \tag{16}$$

for any $\Delta > \delta > 0$, where $d(x, J_t) = \inf_y \{\|x - y\|, y \in J_t\}$ and $\zeta_u(\cdot)$ denotes the gradient of $\zeta(\cdot)$. J_t is the roots set of regression functions and $\zeta_u(\cdot)$. Further, $\zeta(J_t) \triangleq \{\zeta(x): x \in J_t\}$ is nowhere dense, and there exists a constant $c_0 > 0$ such that $\zeta(0) < \inf_{\|u\|=c_0} \zeta(u)$.

2. Along with the subscripts $\{n_k\}$ of any convergent subsequences $\{u_{n_k}\}$

$$\lim_{T\to 0} \limsup_{k\to\infty} \frac{1}{T} \left| \sum_{i=n_k}^{m(n_k,T)} a_i w_i(t) \right| = 0 \quad \text{a.s.} \tag{17}$$

$$\lim_{T\to 0} \limsup_{k\to\infty} \frac{1}{T} \left| \sum_{i=n_k}^{m(n_k,T)} a_i \varphi_i(t) \right| = 0. \quad \text{a.s.} \tag{18}$$

where $m(k, T) \triangleq \max\{m: \sum_{i=k}^{m} a_i \le T\}$, $T > 0$.

3. The regression function $L_{t,k}(u)$ is measurable and uniformly local bounded,

$$\sup_k \sup_{\|u\|<c} \|L_{t,k}(u)\| < \infty, \quad \forall c > 0$$

Then, $d(u_k(t), J_t) \underset{k\to\infty}{\longrightarrow} 0$ a.s.

Remark 4: It deserves to be specially noted that, in (i) of Lemma 3, it is only required the existence of Lyapunov function $\zeta(\cdot)$. That is, nothing about its concrete expression is required to be known. Furthermore, $\zeta(\cdot)$ need not be unique and identical for all t. For the sake of brevity, we shall always use $\zeta(\cdot)$ to denote the corresponding Lyapunov function.

Lemma 4: Assume A5 holds, then (17) holds for $t \in [0, T]$.

Proof: By A5, it is clear that $\sum_{k=1}^{\infty} \mathbb{E}(a_k w_k(t))^2 = R_t \sum_{k=1}^{\infty} a_k^2 < \infty$ a.s., $\forall t \in [0, T]$. This further implies that $\sum_{k=1}^{\infty} a_k w_k(t) < \infty$ a.s. by Khintchine–Kolmogorov Convergence Theorem [30].

By Kronecker Lemma, it follows that

$$a_n \sum_{k=1}^{n} w_k(t) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{a.s.} \tag{19}$$

To prove (17) it suffices to show that $\forall T > 0$, $\delta > 0$, there exists $N > 0$ such that for any $k \ge N$ and for any m: $n_k \le m \le m(n_k, T) + 1$

$$\left\| \sum_{i=n_k}^{m} a_i w_i(t) \right\| \le \delta T \quad \text{a.s.} \tag{20}$$

By noting that $a_k = 1/k$ it is obvious that $a_k - a_{k+1} = a_k a_{k+1}$. By a partial summation and (19), we have that there exists N such that for any $k \ge N$ and for any m: $n_k \le m \le m(n_k, T) + 1$

$$\begin{aligned} \left\| \sum_{i=n_k}^{m} a_i w_i(t) \right\| &= \left\| a_m \sum_{i=1}^{m} w_i(t) - a_{n_k} \sum_{i=1}^{n_k - 1} w_i(t) \right. \\ &\quad \left. + \sum_{i=n_k}^{m-1} (a_i - a_{i+1}) \sum_{j=1}^{i} w_i(t) \right\| \\ &\le \frac{1}{2}\delta T + \sum_{i=n_k}^{m-1} a_{i+1} \left\| a_i \sum_{j=1}^{i} w_i(t) \right\| \\ &\le \frac{1}{2}\delta T + \frac{1}{2}\delta T = \delta T \end{aligned}$$

This completes the proof. □

Theorem 2: Consider system (1) and tracking performance index (4) and assume A1–A7 hold. Then the control sequence $\{u_k(t)\}$ given by ILC algorithm (12) and (13) is bounded a.s. and optimal.

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Proof: According to Theorem 1, we only need to show that $\{u_k(t)\}$ is bounded and $\delta v_k(t) \underset{k\to\infty}{\longrightarrow} 0$. Mathematical induction is used in the following proof.

Firstly consider the case $t = 0$, and we now verify the conditions of Lemma 3.

By Lemma 2 and A6–A7, $\boldsymbol{c}^+\boldsymbol{b}_k(0)$ is bounded. Notice that $\mathcal{D}(u_d(0)) - \mathcal{D}(u)$ is measurable and local bounded, hence (iii) of Lemma 3 is valid.

By A5 and Lemma 4, (17) holds. By A7, both τ_0 and ς_0 equal 0. Thus by A4 and A6, $\|\delta f_k(0)\| \underset{k\to\infty}{\longrightarrow} 0$, $\|\delta \boldsymbol{b}_k(0)\| \underset{k\to\infty}{\longrightarrow} 0$, which is equivalent to that $\varphi_k(0) \underset{k\to\infty}{\longrightarrow} 0$ asymptotically. That is, (18) holds for $t = 0$, and hence (ii) of Lemma 3 is valid.

Now there is only (i) of Lemma 3 left to be verified. Define the Lyapunov function $\zeta(u) \triangleq (u_d(0) - u)^2$, then the root set of $\zeta_u(\cdot)$ and $L_{0,k}(\cdot)$ is

$$J_0 = \begin{cases} u_d(0) & \text{if } u_d(0) \notin I_m \\ I_m & \text{if } u_d(0) \in I_m \end{cases}$$

For this, we take the case $u_d(0) \notin I_m$ and the case $u_d(0) \in I_m$ into account, respectively.

If $u_d(0) \notin I_m$, without loss of generality, let $u_d(0) \in I_r$. Then the left side of (16) comes to

$$\begin{aligned} &\sup_k \sup_{\delta\le d(x,J_t)\le\Delta} \zeta_u(u)L_{t,k}(u) \\ &= \sup_k \sup_{\delta\le d(x,J_t)\le\Delta} -2c^+b_k(t)[u_d(0) - u][\mathcal{D}(u_d(t)) - \mathcal{D}(u)] \end{aligned}$$

It is obvious that $[u_d(0) - u][\mathcal{D}(u_d(t)) - \mathcal{D}(u)] > 0$ whenever $u \neq u_d(0)$, while $\boldsymbol{c}^+\boldsymbol{b}_k(0)$ is positive and tends to some positive constant as $k \to \infty$. Thus (16) is true. Moreover, because $\zeta(0) = (u_d(0))^2$ and $\zeta(u) \to \infty$ as $k \to \infty$, there exists some appropriate c_0 such that $\zeta(0) < \inf_{\|u\|=c_0} \zeta(u)$. Then by Lemma 3, we find $d(u_k(0), J_0) \underset{k\to\infty}{\longrightarrow} 0$, which furthermore implies $u_k(0) \underset{k\to\infty}{\longrightarrow} u_d(0)$ since $J_0 = \{u_d(0)\}$. It is easy to see now that $\{u_k(t)\}$ is bounded and $\delta v_k(0) \underset{k\to\infty}{\longrightarrow} 0$ because of the continuity of the dead-zone function $\mathcal{D}(\cdot)$.

If $u_d(0) \in I_m$, then $\mathcal{D}(u_d(0)) = 0$. Notice that

$$\zeta_u(u)L_{t,k}(u) = \begin{cases} 2\boldsymbol{c}^+\boldsymbol{b}_k(0)g_r(u)(u_d(0) - u), & u \in I_r \\ 0, & u \in I_m \\ 2\boldsymbol{c}^+\boldsymbol{b}_k(0)g_l(u)(u_d(0) - u), & u \in I_l \end{cases}$$

which implies that (16) still holds when $u \notin J_0$. It is easy to verify the rest of (i) of Lemma 3 following the same steps as the case of $u_d(0) \notin I_m$. By Lemma 3 again, $d(u_k(0), I_m) \underset{k\to\infty}{\longrightarrow} 0$, which yields boundedness of $\{u_k(0)\}$ and $\delta v_k(0) \underset{k\to\infty}{\longrightarrow} 0$. From the arguments above, we find the theorem is valid for $t = 0$.

We complete the proof by induction. Now assume the theorem is valid for $s = 0, 1, \ldots, t-1$. Then by the induction assumption and Lemma 2, $\|\delta f_k(s)\| \underset{k\to\infty}{\longrightarrow} 0$, $\|\delta \boldsymbol{b}_k(s)\| \underset{k\to\infty}{\longrightarrow} 0$, $s = 0, 1, \ldots, t$, which means $\varphi_k(t) \underset{k\to\infty}{\longrightarrow} 0$ a.s., that is (18) has been verified. The following steps are exactly similar to the case of $t = 0$. The induction is completed. □

The following corollary is easy to be proved.

Corollary 1: Consider system (1) with $w_k(t) = 0$, $t \in [0, T]$, $k = 1, 2, \ldots$, and assume A1–A7 except A5 hold then $y_k(t) \underset{k\to\infty}{\longrightarrow} y_d(t)$, $\forall t \in [0, T]$, a.s.

Remark 5: From Theorem 2, it can be seen that although the state delays are uncertain and time varying, convergence of the proposed ILC algorithm (12), (13) are still valid. Generally speaking, the proposed ILC algorithm may track the desired output without any effect of state delays. This may indicate that the state delays do not play a significant role in ILC.

Remark 6: From Remark 5 and the proof of Theorem 2, the above results can be easily extended to a class of non-linear systems with multiple time delays, which is described by

$$\begin{cases} x_k(t+1) = f(t, x_k(t-\tau_{t1}), \ldots, x_k(t-\tau_{tm})) \\ \quad + \boldsymbol{b}(t, x_k(t-\varsigma_{t1}), \ldots, x_k(t-\varsigma_{tm}))u_k(t) \\ y_k(t) = \boldsymbol{c}(t)x_k(t) + w_k(t) \end{cases} \tag{21}$$

The corresponding continuous-time case has been discussed in [7, 8] and GLC of non-linear function is required there. As a special case of (21), the corresponding continuous-time case of the following system has been discussed in [10, 11]

$$\begin{cases} x_k(t+1) = \sum_{i=1}^{l} A_i x_k(t-\tau_{ti}) + Bu_k(t) \\ y_k(t) = Cx_k(t) \end{cases} \tag{22}$$

Remark 7: From the proof of Theorem 2, we can see that strictly increasing character of $g_l(\cdot)$ and $g_r(\cdot)$ can be relaxed. Roughly speaking, the dead-zone non-linearity can be replaced by any monotonic increasing divergent functions. The case of monotonic increasing non-linearity without divergence, for instance saturation non-linearity, will be discussed in a coming paper.

4 Numerical illustrations

Consider the following non-linear stochastic system

$$v_k(t) = \mathcal{D}(u_k(t))$$

$$x_k^{(1)}(t+1) = 0.3x_k^{(1)}(t-\tau_t) + 0.2\sin(x_k^{(2)}(t-\zeta_t)) - 0.75v_k(t)$$

$$x_k^{(2)}(t+1) = 0.2\cos(x_k^{(1)}(t-\tau_t)) + 0.3x_k^{(2)}(t-\zeta_t) + 1.3v_k(t)$$

$$y_k(t) = x_k^{(1)}(t) + x_k^{(2)}(t) + w_k(t)$$

where $w_k(t)$, $k = 1, 2, \ldots$, are random variables with normal distribution $\mathcal{N}(0, 0.1^2)$, $t = [1, \ldots, 9]$. Time delays τ_t and ζ_t are assumed that $\tau_1 = \zeta_1 = 0$ and $\tau_t = \zeta_t = 1$, $t = 2, \ldots, 9$. Let the dead-zone function be time invariant

$$\mathcal{D}(u) = \begin{cases} 2x + \sin\left(\dfrac{x-1}{10}\right) - 2, & x > 1 \\ 0, & -1 \le x \le 1 \\ 2x + \sin\left(\dfrac{x+1}{10}\right) + 2, & x < -1 \end{cases}$$

Let the reference trajectory be $y_d(t) = (1/3)t(6 - 0.01t)$, $t = [1, \ldots, 9]$, and let the initial control $u_0(t) \equiv 0$, $\forall t \in [0, 10]$.

In the algorithm we set

$$a_k = \frac{1}{k+1}, \quad M_k = 3^k$$

For each $t \in [1, 9]$ the algorithm runs 500 cycles. The convergence of input for time $t = 1, \ldots, 8$ is show in Fig. 2, and the corresponding tracking errors for time $t = 2, \ldots, 9$ are shown in Fig. 3.

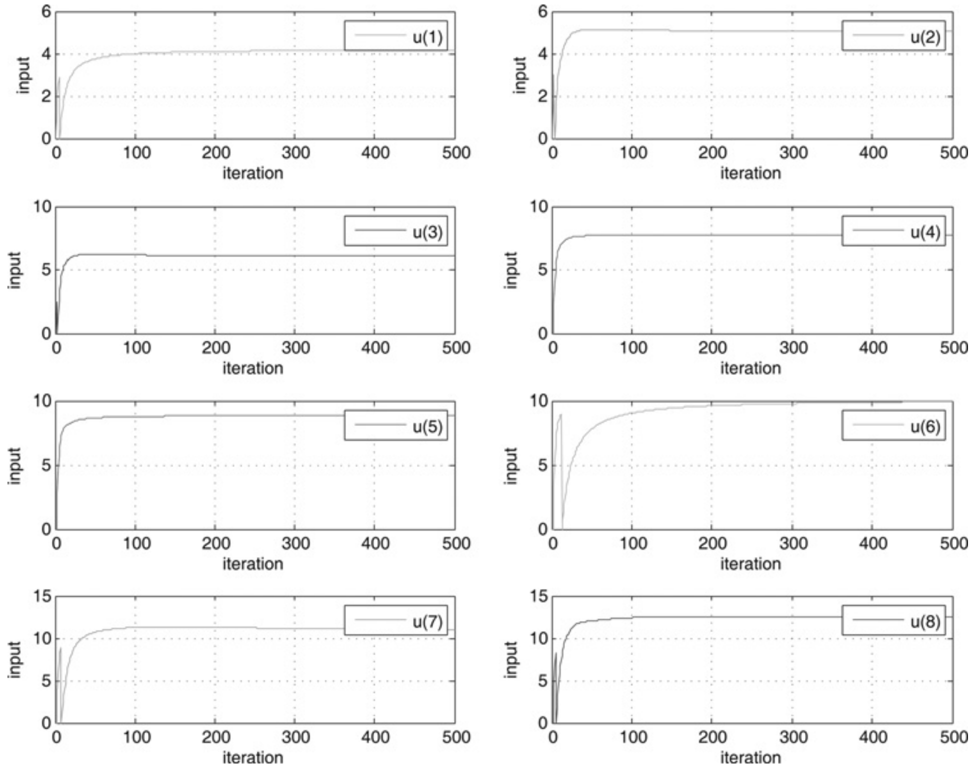

Fig. 2 *Inputs $u_k(t)$ for $t = 1, \ldots, 8$*

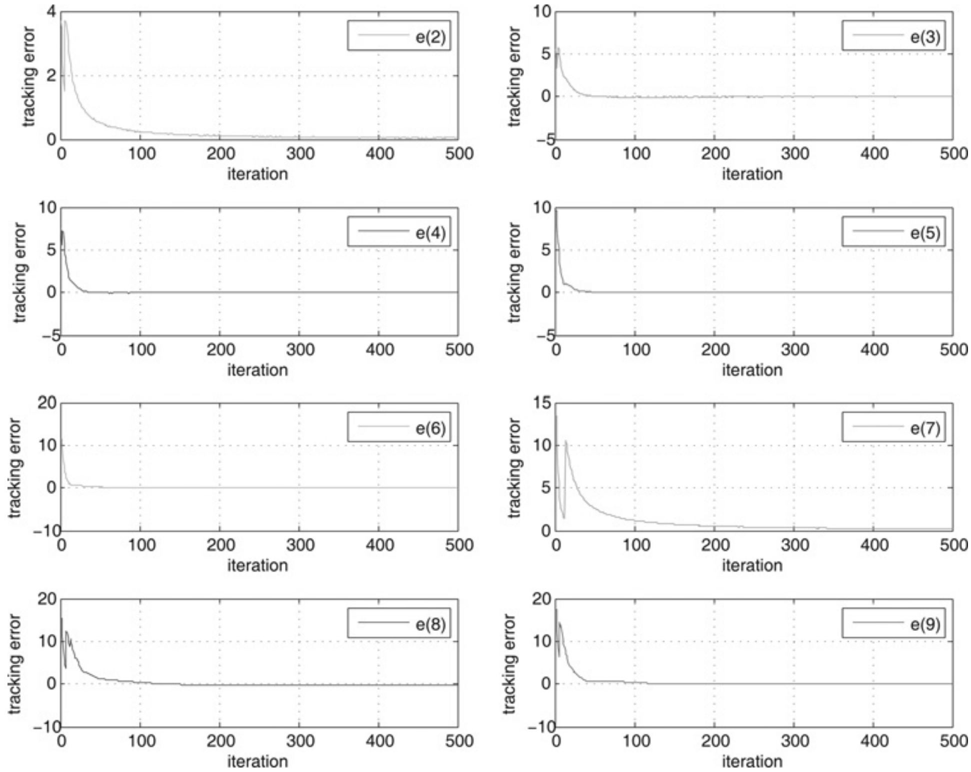

Fig. 3 *Tracking errors $e_k(t)$ for $t = 2, \ldots, 9$*

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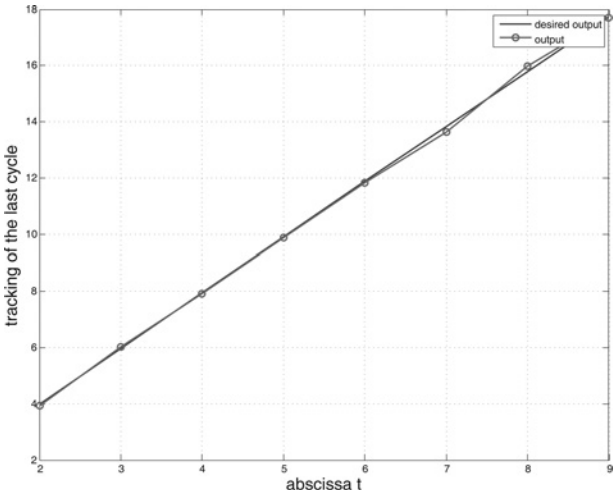

Fig. 4 *Comparison of $y_{500}(t)$ with $y_d(t)$ for $t = 2, \ldots, 9$*

The output trajectory (solid line with circles) at the last (i.e. the 500th) cycle and the reference trajectory (solid line) are put together for a comparison in Fig. 4.

5 Conclusions

The ILC for non-linear systems with time delays and dead-zone input in presence of measurement noise is considered in this paper. We construct the ILC algorithm based on stochastic approximation, and prove that the control sequence is bounded and optimal. Our work has several features different from previous works. First, no GLC is required on non-linear functions; second, the dead-zone is described by a general form, which is taken less into account; third, time delays could be time varying and measurement noise is corrupted.

For future research it is of interest to consider stochastic delays in state, which may be a big challenge. Also, our paper focuses on SISO system, the corresponding results of multi-input-multi-output (MIMO) system are much attractive.

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