Systems & Control Letters 77 (2015) 30–39

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

ILC for networked nonlinear systems with unknown control direction through random Lossy channel[☆]

Dong Shen [*], Youqing Wang

College of Information Science and Technology, Beijing University of Chemical Technology, Beijing 100029, PR China

ARTICLE INFO

Article history:
Received 4 July 2014
Received in revised form
15 December 2014
Accepted 22 December 2014

Keywords:
Iterative learning control
Networked control system
Random packet losses
Stochastic approximation

ABSTRACT

The iterative learning control is constructed for the discrete-time networked nonlinear systems with random measurement losses and unknown control direction, which have not been studied simultaneously in literature. Differing from the conventional Bernoulli random variable model, the random packet loss is modeled by an arbitrary stochastic sequence with bounded length requirement, which is a new model of realistic packet losses. A novel regulating approach based on truncations is introduced to make the proposed algorithm find the correct control direction adaptively, and then guarantee the almost sure convergence property. Therefore, this paper has three major innovations compared with reported studies, namely, the stochastic sequence model of packet loss, the novel control direction regulation method, and the almost sure convergence property of the proposed algorithm. An illustrative example shows the effectiveness of the proposed approach.

1. Introduction

In our daily lives, it appears that we would perform better and better if we could repeat some task. This is the basic idea of iterative learning control (ILC), which has been widely applied in many systems that could perform the same task repeatedly, such as chemical process, hard disk drives, and robotics. In such systems the input and output information generated in the previous batches are of great help for improving the performance, if they are used to design control signal for the next batch. Since proposed in the 1980s, ILC has been discussed from multiple point of views as it could realize accurate tracking with simple control algorithm [1–4]. Moreover, networked control systems are widely applied as they own much more flexibility dealing with complex system. However, in networked control systems, the data packets are not always transmitted successfully in consequence of network congestion, linkage interrupt, transmission error, and etc., and thus the system performance may be reduced. Furthermore, it is required to take the control direction as prior information when designing the ILC algorithm. Here the so-called control direction is the multiplier of the control term in the dynamic system equation, representing the motion directions of the system under any given control. This motivates us to consider ILC for networked nonlinear system with random packet losses and unknown control direction.

The publications related to iterative learning control for networked control systems, where the information packets might be random lost, are very rare, although there have been quite a few of discussions in conventional networked control systems. In most papers, the packet loss is modeled as a Bernoulli random variable, whose value is 1 when the packet is successfully transmitted and 0 otherwise. Ahn et al. and Bu et al. made many contributions to this topic from different perspectives. Ahn et al. studied this topic for MIMO time-invariant systems and showed the mean-square stability under data dropouts by the Kalman filtering based techniques, which is proposed in [5], for iterative learning control [6–8]. The major differences among these papers are the packet loss locations. In particular, in [6] only the measurement output was assumed to be randomly lost when transmitting back to control center. Besides, the output vector was assumed to be completely lost if packet loss occurs. In practical system, maybe only part of the multi-dimensional output is lost but the other part is transmitted back. The analysis of this case was considered in [7]. [8] further discussed the case that packet loss happened to the control signals as well as output signals. Bu et al. considered iterative learning control for networked control systems from the statistics point of view [9–12], where the main idea is transforming the stochastic equation/equality into deterministic equation/equality by taking mathematical expectations and then showing the stability/

☆ This work was supported by National Natural Science Foundation of China (61304085 & 61374099), Beijing Natural Science Foundation (4152040), the Fundamental Research Funds for the Central Universities (ZY1318), and Beijing Higher Education Young Elite Teacher Project (YETP0505).

* Corresponding author.
E-mail addresses: shendong@mail.buct.edu.cn (D. Shen), wang.youqing@ieee.org (Y. Wang).

http://dx.doi.org/10.1016/j.sysconle.2014.12.008
0167-6911/

convergence conditions. In [9], the linear time-invariant discrete system was lifted into the so-called super-vector form, consequently the model turned into 1-D form which only evolved along the iteration axis. Then the iteration equation of tracking errors was directly obtained. The stability analysis was given in [9] based on the equation and exponential stability for asynchronous dynamical systems [13]. Another stability result was given in [10] for SISO linear time-invariant system under data dropouts, where mathematical expectations are taken to both sides of the iteration equation of tracking errors directly, and then gave the stability condition according to the mathematical expectation of tracking error. Both [11,12] considered the nonlinear system case. The regression equation was first established, and then regression inequality was given by taking expectations and contract mapping method, whence the effect of stochastic data dropout was eliminated so that the convergence condition was provided. The main difference between [11] and [12] is the selection of update information. Specifically, in [11], the algorithm would not update if the data is lost; while in [12], the algorithm would use previous data if no new data arrives. The differences between [11,12] and this note are as follows: the packet loss is modeled by Bernoulli sequence in [11,12], while it is described by a stochastic sequence without obeying any distribution in this paper; the convergence of proposed algorithms is proved in the mathematical expectation meaning in [11,12], while it is in almost sure meaning in this paper; all nonlinear functions are assumed globally Lipschitz in [11,12], while in this paper this requirement has been removed; furthermore, random measurement noises are taken into account in this paper which are not considered in [11,12].

Combining contributions of Ahn et al. and Bu et al., one can find the following facts on iterative learning control for networked control system with random packet loss: (a) the convergence is in mean-square sense or mathematical expectation sense; (b) the random packet loss is modeled by a binary Bernoulli random variable; and (c) none stochastic noise is involved expect [7]. However, it is found that the packet loss is normally modeled in three kinds of formulations, which are Bernoulli random variable, arbitrary stochastic sequence, and Markov chain, respectively, by observing the physical phenomenons of packet losses. The latter two cases gain little attention, which motivates this study. Specifically speaking, in this paper, the random packet loss is considered as an arbitrary stochastic sequence with suitable conditions, which differs from the existing formulation. The Bernoulli random variable model has been studied a lot in previous publications, while few results have been gained for the arbitrary stochastic sequence model without obeying any probability distribution. As in many practical system, any kind of probability distribution is a factitious presupposition of packet loss. Thus a black-box model would be more suitable, since one knows nothing about the packet loss in advance. It is also noted that the techniques used in Ahn's and Bu's papers for the Bernoulli process could not be applied for this case. Therefore, it is interesting to consider the arbitrary stochastic sequence model of packet loss.

It is difficult to find a publication in the field of ILC when the control direction is unknown. The technique of Nussbaum-type gain [14] is the most common method in literature, where the Nussbaum-type gain plays a role as control direction probe. ILC is designed in [15,16] for SISO learning control systems based on this technique. A class of affine nonlinear systems is considered in [15], where the system nonlinearities are assumed bounded. It is shown that the tracking error would converge to a desired bounded domain. Xu et al. studies a class of parameterized nonlinear systems with known nonlinear functions but unknown parameters [16]. It is proved that the tracking error converges to zero in the sense of L_T^2. However, the systems in [15,16] are continuous-time deterministic systems to facilitate the application of Nussbaum-type gain. The techniques are not suitable for discrete-time systems or stochastic systems. In [17], Chen gives a novel ILC algorithm for stochastic discrete-time systems without any information about input/output coupling matrices based on Kiefer–Wolfowitz algorithm with random differences [18], and show that the algorithm would converge to the optimal control with probability one. However, it is inappropriate to use this approach in our problem due to the existence of random packet loss. In [19] the linear stochastic system is considered and a novel data-based switch mechanism for direction probing is proposed by using the accumulated tracking performance. But it may be hard to be applied to nonlinear system, which requires further efforts. In this paper, the unknown control direction is handled by using the truncation number of an improved stochastic approximation algorithm [18]. This approach is simple but effective for nonlinear stochastic system.

In this study, ILC is considered for networked nonlinear system with random packet loss and unknown control direction. Concerning networked system we mean the plant and controller are connected by networks, where packets may be lost during transmission. For problem clear, only the measurement loss is taken into account, but it is effortless to extend to the general case, namely, both measurement and control packet losses are considered. A random set is introduced to denote the loss locations at each iteration, then the measurement loss is modeled by an arbitrary stochastic sequence with some bounded length requirement. It is worth pointing out that the stochastic sequence model could not be covered by the traditional Bernoulli random variable model, and vice versa. In order to deal with the unknown direction, a novel regulating approach is proposed to make the algorithm find its correct control direction adaptively. The almost sure convergence analysis is also given. Thus our study makes three major differences with previous studies, namely, the stochastic sequence model of packet loss, the novel control direction regulation method, and the almost sure convergence property of the proposed algorithm.

The rest of the paper are arranged as follows: Section 2 provides the system formulation and the model of packet loss, as well as preliminary lemma; Section 3 gives the ILC algorithm and the almost sure convergence analysis, where the effectiveness of the proposed direction switch approach also is shown; Section 4 provides an illustrative example to verify the theoretical analysis; some concluding remarks are given in Section 5.

2. Problem formulation

Consider the following SISO affine nonlinear discrete-time system

$$\begin{aligned} x_k(t+1) &= f(t, x_k(t)) + \mathbf{b}(t, x_k(t))u_k(t) \\ y_k(t) &= \mathbf{c}(t)x_k(t) + w_k(t) \end{aligned} \tag{1}$$

where $t \in \{0, 1, \dots, N\}$ denotes the time instant in an iteration of the process, while $k = 1, 2, \dots$ label different iterations. $u_k(t) \in \mathbb{R}$, $x_k(t) \in \mathbb{R}^n$, and $y_k(t) \in \mathbb{R}$ denote the input, state, and output, respectively. $f(t, x_k(t))$, $\mathbf{b}(t, x_k(t))$, and $\mathbf{c}(t)$ denote unknown system information. $w_k(t)$ is the measurement noise.

The setup of the control system is illustrated as in Fig. 1. For convenience, only measurement loss is considered in this short paper, which could be easily extended to the case that both measurement and control signals are lossy. As shown in Fig. 1, the measurement signals are transmitted back through a lossy channel, which could be regarded as a switch that opens and closes in a random manner. The rest of the block diagram follows conventional ILC framework.

In previous studies, the random packet loss is modeled as a binary Bernoulli random variable. However, in this paper, a random sequence model of the random packet loss is proposed. To this end, denote $\mathcal{M}_k$ as the random set of time locations at which measurements are lost in the k-th iteration. In other words, $t_0 \in \mathcal{M}_k$ if $y_k(t_0)$ is lost.

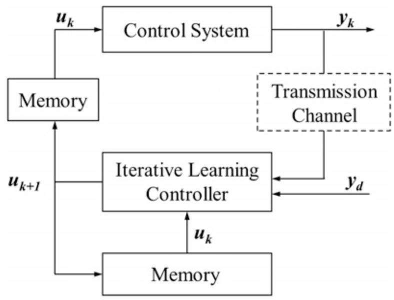

Fig. 1. Block diagram of networked control system with measurement packet loss.

Let $\mathcal{F}_k \triangleq \sigma\{y_j(t), x_j(t), w_j(t), v_j(t), 0 \le j \le k, t \in \{0, \dots, N\}\}$ be the σ-algebra generated by $y_j(t), x_j(t), w_j(t), v_j(t), 0 \le t \le N, 0 \le j \le k$. Then the set of admissible controls is defined as

$$U = \{u_{k+1}(t) \in \mathcal{F}_k, \sup_k \|u_k(t)\| < \infty, \text{ a.s. } t \in \{0, \dots, N-1\}, k = 0, 1, 2, \dots\}.$$

The control purpose is to find the control sequences

$$\{u_k(t), k = 0, 1, 2, \dots\} \subset U$$

under random packet loss environment to minimize the super limitation of the averaged tracking errors $\forall t \in \{0, 1, \dots, N\}$

$$V(t) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \|y_k(t) - y_d(t)\|^2 \tag{2}$$

where $y_d(t), t \in \{0, 1, \dots, N\}$ is the tracking target.

The following assumptions are used.

A 1. The tracking target $y_d(t)$ is realizable in the sense that there exist $u_d(t)$ and $x_d(0)$ such that

$$\begin{aligned} x_d(t+1) &= f(t, x_d(t)) + \mathbf{b}(t, x_d(t))u_d(t) \\ y_d(t) &= \mathbf{c}(t)x_d(t). \end{aligned} \tag{3}$$

A 2. The functions $f(\cdot, \cdot)$ and $\mathbf{b}(\cdot, \cdot)$ are continuous with respect to the second argument.

A2 could be relaxed to the case that the functions $f(t, x)$ and $\mathbf{b}(t, x)$ have discontinuities with respect to x away from $x_d(t)$. Since $x = x_d(t)$ is unknown, thus A2 is simply assumed.

A 3. The input/output coupling value $\mathbf{c}(t+1)\mathbf{b}(t, x_k(t))$ is unknown, but it is nonzero and does not change its sign. Without loss of any generality, it is assumed $\mathbf{c}(t+1)\mathbf{b}(t, x_k(t)) > 0$.

A3 is the assumption on control direction. As the SISO model is taken into account, the control direction actually is the sign of the coupling value, i.e., the control direction is $+1$ or -1. However, this case can be extended to the MIMO case by using similar techniques of [20].

A 4. The initial values can asymptotically be precisely reset in the sense that $x_k(0) \to x_d(0)$ as $k \to \infty$.

A4 is the re-initialization condition. In many papers the initial state usually is required to be $x_d(0)$ [2,21], while in this paper it is only required that the accurate initial state is reset asymptotically. Besides, it is an interesting topic to study how to realize the asymptotical re-initialization condition by suitable design learning algorithm of initial values [22,23].

A 5. For each t the measurement noise $\{w_k(t)\}$ is a sequence of independent and identically distributed random variables with $\mathbb{E}w_k(t) = 0$, $\sup_k \mathbb{E}w_k^2(t) < \infty$, and $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} w_k^2(t) = R_w^t$, a.s., where R_w^2 is unknown.

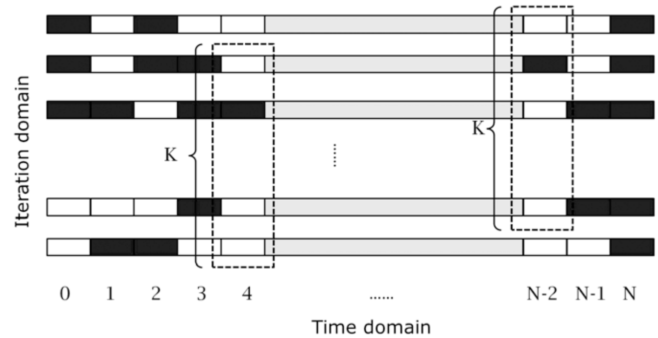

Fig. 2. Diagram of packet losses.

In A5 the noise assumption is made according to the iteration axis rather than the time axis. This requirement is not rigorous as the process would be performed repeatedly.

A 6. For each t, the measurement packet loss is random without obeying any certain probability distribution, but there is a number K such that during successive K iterations, at least in one iteration the measurement is successfully sent back.

Fig. 2 is given to explain A6, where a horizontal bar denotes an iteration process. In any bar, the white rectangle and black rectangle denote the lost packet and successfully transmitted packet, respectively. The gray part of each horizontal bar denotes the omission part. The meaning of A6 is that for the time domain at certain time t, one packet at this time should be received at least once for successive iterations K, as shown in Fig. 2. Taken the time instance $t = 4$ as an illustration, A6 means that there is at least one black rectangle for any successive K horizontal bars. The number K in A6 is not necessary to be known prior, in other words, only the existence of such number is required. Thus this condition means that the measurements should not be lost too much to guarantee the convergence in almost sure sense. It is worth pointing out that this model of packet loss is different from the traditional binary Bernoulli one, which could not be covered by A6 and vice versa.

Remark 1. The value of K is a reflection of the rate of packet loss. Roughly speaking, a larger K corresponds to a higher rate of packet loss, while a smaller K corresponds to a lower rate of packet loss. Thus the convergence speed usually is faster if the number K is smaller since most packets are successfully transmitted, and vice versa. However, as one could see, K is the worst case of packet loss, which means that the packet loss at a certain time could only happen at most $K-1$ successive iterations. Thus it is not an accurate representation of the rate of packet loss. While in the traditional binary Bernoulli model, the probability of packet loss should be known accurately.

Remark 2. It should be pointed out that A6 hints that the packet loss is assumed not totally stochastic in this paper. As a matter of fact, the limitation on randomness is required because of technical demand and one could find that the algorithm also behaves well under Bernoulli case of packet losses from the illustrative simulations. It is of great interest to relax this limitation in the future. Besides, only the existence of K rather than its exact value is required in this study, as one could see from the algorithms (4)–(6) proposed later in Section 3. Thus, from this point of view, A6 is not too restrictive for practical applications.

For simplicity of writing let us set $f_k(t) = f(t, x_k(t))$, $f_d(t) = f(t, x_d(t))$, $\mathbf{b}_k(t) = \mathbf{b}(t, x_k(t))$, $\mathbf{b}_d(t) = \mathbf{b}(t, x_d(t))$, $e_k(t) = y_d(t) - y_k(t)$, $\delta x_k(t) = x_d(t) - x_k(t)$, $\delta u_k(t) = u_d(t) - u_k(t)$, $\delta f_k(t) = f_d(t) - f_k(t)$, $\delta\mathbf{b}_k(t) = \mathbf{b}_d(t) - \mathbf{b}_k(t)$, $\mathbf{c}^+ f_k(t) = \mathbf{c}(t+1)f(t, x_k(t))$, $\mathbf{c}^+\mathbf{b}_k(t) = \mathbf{c}(t+1)\mathbf{b}(t, x_k(t))$.

Now the following lemma on continuous of nonlinear functions is given, whose proof is put in the Appendix.

Lemma 1. *Assume A1–A6 hold for system (1). If* $\lim_{k\to\infty}\delta u_k(s)=0$, $s=0,1,\ldots,t$, *then at time* $t+1$, $\|\delta x_k(t+1)\|\xrightarrow[k\to\infty]{}0$, $\|\delta f_k(t+1)\|\xrightarrow[k\to\infty]{}0$, $\|\delta \mathbf{b}_k(t+1)\|\xrightarrow[k\to\infty]{}0$.

Theorem 1. *For the nonlinear system (1) and tracking objective* $y_d(t)$, *assume A1–A6 hold, then the index (2) will be minimized for any arbitrary time t if the control sequence* $\{u_k(s)\}$ *is admissible and satisfies* $u_k(s)\xrightarrow[k\to\infty]{}u_d(s)$, $s=0,1,\ldots,t-1$. *In this case,* $\{u_k(s)\}$ *is called the optimal control sequence.*

Proof. According to A5 and the definition of $\mathcal{F}_k$, it follows that $\mathcal{F}_k$ is independent of $\{w_l(t), l=k+i, i=1,2,\ldots,\forall t\}$, thus $\{w_k(t),\mathcal{F}_k\}$ is a martingale difference sequence. Meanwhile, input, output, and state vectors are all adapted to $\mathcal{F}_k$. Therefore by (1) and A1

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\|y_k(t)-y_d(t)\|^2\\
&=\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\|\mathbf{c}(t)(x_k(t)-x_d(t))+w_k(t)\|^2\\
&=\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\|\mathbf{c}(t)\delta x_k(t)\|^2(1+o(1))\\
&\quad+\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\|w_k(t)\|^2\\
&\geq \limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\|w_k(t)\|^2=R_w^t.
\end{aligned}$$

The sufficient and necessary condition to achieve the minimum is

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\|\mathbf{c}(t)\delta x_k(t)\|^2=0$$

which is true when $\mathbf{c}(t)\delta x_k(t)\xrightarrow[k\to\infty]{}0$. While the latter holds if $\delta u_k(s)\xrightarrow[k\to\infty]{}0$, $s=0,1,\ldots,t-1$ by Lemma 1. The proof is completed. □

3. ILC algorithm and its convergence

Now define the ILC leading the performance index (2) to its minimum. As the control direction is unknown, one has to introduce a switch mechanism to make the algorithm find the correct control direction adaptively. Intuitively, the algorithm should track good enough if it chooses the correct control direction and behave badly if it chooses the wrong control direction. This motivates us to define the following update law by using the tracking performance to search the control direction, which is inspired from [24]. Remind that the correct control direction is "+1" in this paper by A3.

$$\bar{u}_{k+1}(t)=u_k(t)+a_k(-1)^{\sigma_k(t)}I_{\{(t+1)\notin\mathcal{M}_k\}}(y_d(t+1)-y_k(t+1)) \quad (4)$$

$$u_{k+1}=\bar{u}_{k+1}I_{\{|\bar{u}_{k+1}|<M_{\sigma_k(t)}\}} \quad (5)$$

$$\sigma_k(t)=\sum_{j=1}^{k-1}I_{\{|\bar{u}_{j+1}|>M_{\sigma_j(t)}\}},\qquad \sigma_0(t)=0 \quad (6)$$

where $\{M_k\}$ is a sequence of positive real numbers such that $M_{k+1}>M_k$, $\forall k$ and $M_k\xrightarrow[k\to\infty]{}\infty$. Besides, a_k is the decreasing gain such that $a_k>0$, $a_k\to 0$, $\sum_{k=0}^{\infty}a_k=\infty$, $\sum_{k=0}^{\infty}a_k^2<\infty$, and $I_{\{\text{event}\}}$ is an indicator function meaning that it equals 1 if the event indicated in the bracket is fulfilled, and 0 if the event does not hold. The initial input $u_k(t)$ is simply valued zero.

Remark 3. The number of truncations $\sigma_k(t)$ is used to regulate the control direction in the algorithm. Specifically, the term $(-1)^{\sigma_k(t)}$ is the regulating term, and "+1" is the correct direction. If the wrong direction "−1" is applied in the algorithm, it will be proved that the algorithm diverges and forces the truncation mechanism to work (see Lemma 4 for details). The underlining reason is the formulation of the learning step-length a_k. The divergence of the algorithm with wrong direction could be seen from the numerical illustration. As results, the direction shifts to the correct "+1".

Remark 4. In this paper, noticing the term $I_{\{(t+1)\notin\mathcal{M}_k\}}$ in (4), the algorithm would not update if the corresponding measurement packet is lost. However, in practice, if the corresponding packet is lost in the k-th iteration, the last available packet at the same time from the previous iteration could be used for updating. This is another kind of data selection method. It should be pointed out that the packet loss may happen in successive iterations randomly. This will result in multiple update of input signal with one packet of tracking information. The algorithm would remain convergent if A6 holds. The proof is similar to the followings but more complex. Therefore, a simple data selection mechanism is adopted in this paper. In other words, if the tracking packet is available, the algorithms use this data for updating; while if the tracking packet is lost, then the algorithms would not update until the arrival of the new data.

The following theorem shows the convergence results of the proposed algorithm.

Theorem 2. *For the nonlinear system (1) and index (2), assume A1–A6 hold, then* (c) *the number of truncations of the algorithm (4)–(6) is finite;* (b) *the algorithm (4)–(6) shifts the correct direction when stops switching;* (c) *the input sequence* $\{u_k(t)\}$ *generated by (4)–(6) is optimal.*

In order to make the proof of the theorem concise and clear, some key steps of the proof are made into auxiliary lemmas as follows. The proofs of them are put in the Appendix.

Lemma 2. *Consider the algorithm (4)–(6) without truncations and remove the regulating term* $(-1)^{\sigma_k(t)}$, *then the input sequence generated by the modified algorithm converges to the desired input in (3) for time* $t=0$.

Lemma 3. *Consider the algorithm (4)–(6) without truncations for time* $t=0$, *remove the regulating term* $(-1)^{\sigma_k(0)}$, *and replace* a_k *by* a_{k+m}, *then* $\delta u_k(0)$ *generated by the modified algorithm with any fixed initial value is uniformly bounded for any integer* $m>0$.

Lemma 4. *Consider the algorithm (4)–(6) for time* $t=0$, *then the algorithm stops at the correct direction and converges to the optimal control whenever the algorithm stops truncation.*

Based on the above lemmas, the proof of Theorem 2 can be given now.

Proof. The proof is carried by mathematical induction along the time axis t.

Step 1 *(Base step).* In this step, the validity of the theorem for $t=0$ is proved.

Subtracting the algorithm without truncations

$$u_{k+1}(0)=u_k(0)+a_kI_{\{1\notin\mathcal{M}_k\}}(y_d(1)-y_k(1)) \quad (7)$$

from the term $u_d(0)$ leads to

$$\begin{aligned}
\delta u_{k+1}(0)&=\delta u_k(0)-a_kI_{\{1\notin\mathcal{M}_k\}}(y_d(1)-y_k(1))\\
&=\delta u_k(0)-a_kI_{\{1\notin\mathcal{M}_k\}}\mathbf{c}^+\mathbf{b}_k(0)\delta u_k(0)\\
&\quad-a_kI_{\{1\notin\mathcal{M}_k\}}[\varphi_k(0)-w_k(1)]
\end{aligned}\quad (8)$$

where $\varphi_k(t)$ is defined by

$$\varphi_k(t)=\mathbf{c}^+\delta f_k(t)+\mathbf{c}^+\delta\mathbf{b}_k(t)u_d(t) \quad (9)$$

and $\varphi_k(0)$ is the value of (9) at $t = 0$. Here, $\mathbf{c}^+\delta f_k(t) = \mathbf{c}(t+1)\delta f(t, x_k(t))$, $\mathbf{c}^+\delta\mathbf{b}_k(t) = \mathbf{c}(t+1)\delta\mathbf{b}(t, x_k(t))$.

By Lemma 2, one has $\delta u_k(0) \xrightarrow[k\to\infty]{} 0$ for (8) with any initial value. Therefore, if the algorithm (8) is with initial value $\delta u_0(0) = u_d(0)$, then there is an $L > 0$ such that $\|\delta u_k(0)\| < L$ for $\{\delta u_k(0)\}$ generated by the algorithm. Then by Lemma 3 it is seen that $\{\delta u_k(0)\}$ produced by the following algorithm with any integer $m > 0$ is bounded if the initial value is taken as $\delta u_0(0) = u_d(0)$:

$$\delta u_{k+1}(0) = \delta u_k(0) - a_{k+m} I_{\{1\notin\mathcal{M}_k\}} \mathbf{c}^+\mathbf{b}_k(0)\delta u_k(0) - a_{k+m} I_{\{1\notin\mathcal{M}_k\}} [\varphi_k(0) - w_k(1)]. \quad (10)$$

Without loss of any generality, denote the boundary as L. In other words, for any integer $m > 0$ by taking the initial value $\delta u_0(0) = u_d(0)$, $\{\delta u_k(0)\}$ generated by (10) satisfies $\|\delta u_k(0)\| < L$. After each truncation the algorithm (4)–(6) is pulled back to zero, and hence the control sequence has an upper bound $L' \triangleq L + \|u_d(0)\|$, whenever the algorithm has switched on the correct direction.

If the number of truncations were infinite, then the algorithm would be switched to the correct direction infinitely many times. Let the algorithm be with the correct direction and let k be sufficiently large so that $M_{\sigma_k(0)} > L'$. This implies that the algorithm (4)–(6) will no longer be truncated when the correct direction is switched on since then, which contradicts the assumptions of infinitely many truncations. Therefore, the algorithm (4)–(6) may have only finite number of truncations, and hence the algorithm is bounded. Thus Theorem 2(a) is proved for $t = 0$.

By Lemma 4, the impossibility of stopping at the wrong direction is verified, thus the algorithm stops at the correct direction. Then by Lemma 2 the algorithm converges to the optimal control, which means Theorem 2(b) and (c) hold for $t = 0$.

Step 2 *(Inductive step)*. The cases for the other time instances are proved in this step.

Now assume that the conclusion of the theorem is valid for $0, 1, \ldots, t-1$, we proceed to prove that it is also true for t. By the inductive assumptions it is seen that $u_k(0), u_k(1), \ldots, u_k(t-1)$ are optimal, i.e., $\delta u_k(s) \xrightarrow[k\to\infty]{} 0$, $s = 0, 1, \ldots, t-1$, and then $\varphi_k(s) \xrightarrow[k\to\infty]{} 0$ by Lemma 1. Then completely the same argument as that used for $t = 0$ leads to the desired conclusion. The proof of the theorem is completed. □

Remark 5. In continuous-time systems, using Nussbaum-type gain is a major method to search the control direction. However, in discrete-time systems, it is hard to build a similar gain. As a matter of fact, the intuitive idea for direction switching is that the algorithm would diverge if the wrong direction is selected while the algorithm would converge if the correct direction is selected. The truncation used in this paper is a simple implementation of the above idea.

4. Illustrative simulation

Consider the following affine nonlinear system as an example, where the state is of two dimension,

$$x_k^{(1)}(t+1) = 0.8x_k^{(1)}(t) + 0.3\sin(x_k^{(2)}(t)) + 0.23u_k(t)$$
$$x_k^{(2)}(t+1) = 0.4\cos(x_k^{(1)}(t)) + 0.85x_k^{(2)}(t) + 0.33u_k(t)$$
$$y_k(t) = x_k^{(1)}(t) + x_k^{(2)}(t) + w_k(t)$$

where $x_k^{(1)}(t)$ and $x_k^{(2)}(t)$ denote the first and second dimensions of $x_k(t)$, respectively. It is easy to see that $\mathbf{c}^+\mathbf{b}(t) = 0.23 \times 1 + 0.33 \times 1 = 0.56 > 0$, thus $+1$ is the correct control direction.

For simple illustration, let $N = 40$, and the measurement noise $w_k(t)$ is assumed zero-Gaussian distributed, i.e., $w_k(t) \sim N(0, 0.1^2)$.

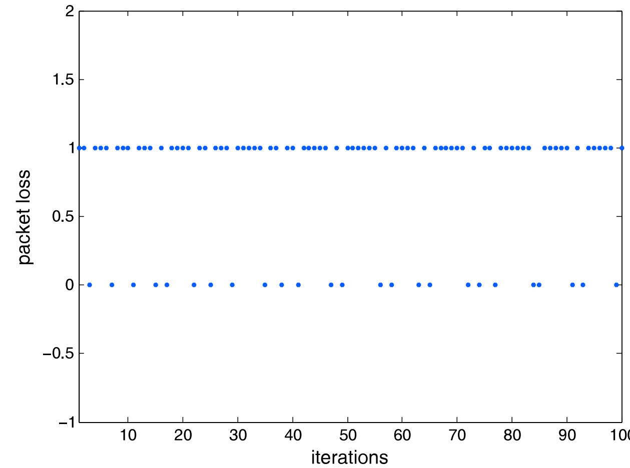

Fig. 3. Illustration of packet loss in the stochastic sequence case for an arbitrary time instance.

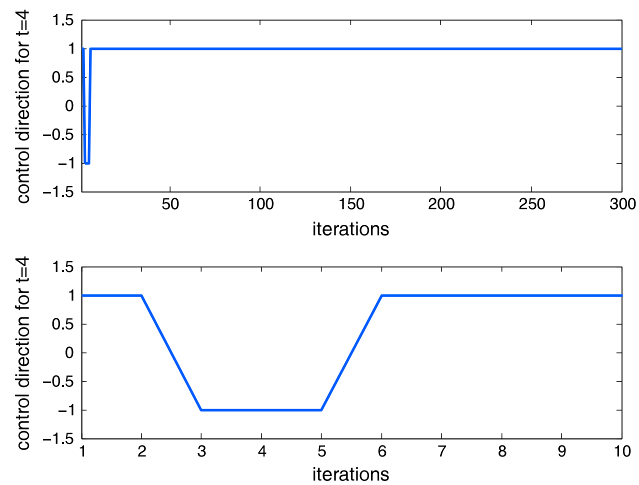

Fig. 4. Control direction regulation at $t = 4$.

In order to simulate measurement loss, first separate iteration steps into groups of four successive iterations, i.e., $\{1, 2, 3, 4\}$, $\{5, 6, 7, 8\}, \ldots$, and then randomly select one iteration from each group for each t, say, $1, 6, 9, \ldots$ for example. For these selected steps the control is not updated, which means the corresponding measurements are lost. Fig. 3 is an arbitrary illustration of packet loss step for the first 100 iterations, where 0 denotes the packet is lost while 1 means that the packet is successfully transmitted.

The reference trajectory is $y_d(t) = 20\sin(\frac{t}{20}\pi)$. The initial control action is simply given as $u_0(t) = 0, \forall t$. The learning gain chooses $a_k = \frac{1}{k+1}$ and the parameter used in the algorithm is $M_k = 4^k$. The algorithm has run 300 iterations.

The control direction regulation is demonstrated for $t = 4$ in Fig. 4. For the example given here the correct control direction is $+1$, and from the figure it is seen that the algorithm switches to the wrong control direction at the 3rd, 4th, and 5th iterations, and switches to the correct control direction since then. This shows that the algorithm could find its correct control direction adaptively. In Fig. 4 the upper figure shows the control direction for the whole 300 cycles, while the lower figure shows the first 10 cycles with enlarged scale.

The tracking result of the 300th iteration is shown in Fig. 5, where the solid line with circle is the reference signal and the dashed line with diamond denotes the actual output $y_{300}(t)$. In our study, the discrete-time case is considered and thus the reference trajectory actually is a set of values rather than a curve. It is noticed

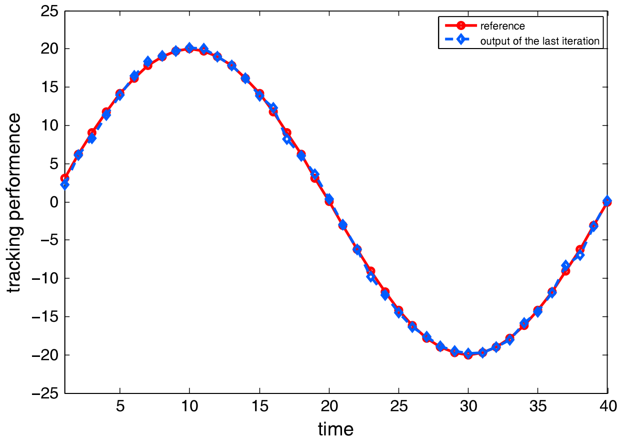

Fig. 5. $y_{300}(t)$ vs. $y_d(t)$.

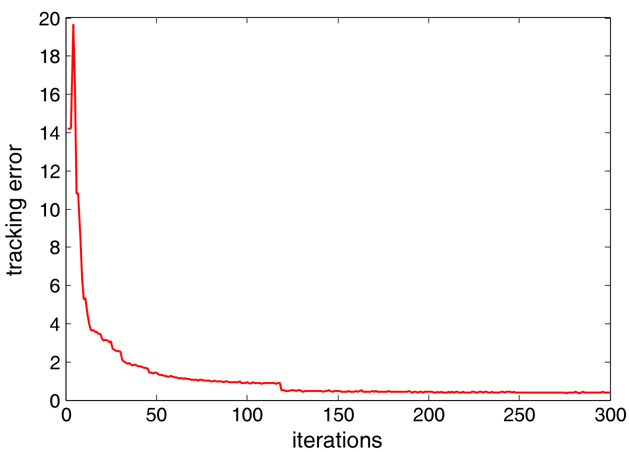

Fig. 6. The average absolute tracking error $\sqrt{\frac{\sum_{t=1}^{T}\|e_k(t)\|^2}{T}}$.

that the output could track the desired positions effectively, which shows the convergence and effectiveness of the proposed algorithms. Besides, the deviations in the figure are caused by stochastic measurement noises, which cannot be canceled by any learning algorithms since the noise is completely unpredictable.

Besides, the averaged absolute tracking error for each iteration is defined as $\sqrt{\frac{\sum_{t=1}^{N}\|e_k(t)\|^2}{N}}$. Noticing that stochastic noise is involved in the above index, thus the defined error could not decrease to zero as the number of iterations goes to infinity. Fig. 6 shows that trajectory behaviors of the defined tracking error.

From the implementation of measurement loss and Fig. 3, one could find that the maximum number of successive packet loss iterations is 2, thus in this case it is noted $K = 3$. In other words, during successive three iterations, at least in one iteration the measurement is successfully sent back. Besides, it is easy to compute that the rate of packet loss is 25%, since only one iteration out of every four iterations misses its measurement data. One may be interested in the influence of K and the loss rate, thus more cases of packet losses should be simulated. To be specific, three more cases are considered: $K = 3$, loss rate = 10%; $K = 3$, loss rate = 50%; and $K = 7$, loss rate = 75%. The averaged absolute tracking errors $\sqrt{\frac{\sum_{t=1}^{N}\|e_k(t)\|^2}{N}}$ for each case are shown in Fig. 7. Noticing Remark 1, it is worthy mentioning that K denotes the worst level of packet loss, and while loss rate denotes the average level of packet loss. As one could see, larger K and/or larger loss rate would result in larger tracking errors.

The simulations are also made for the case that the random packet loss is modeled as a binary Bernoulli random variable γ, which is distributed as $P(\gamma = 1) = 0.75$ and $P(\gamma = 0) = 0.25$, as illustrated in Fig. 8. As can be seen, in this case, the length requirement may be not satisfied. The simulation results are similar to Figs. 3–6, as shown in Figs. 9–11, which imply that the proposed algorithm has a good and robust performance for different random packet loss forms.

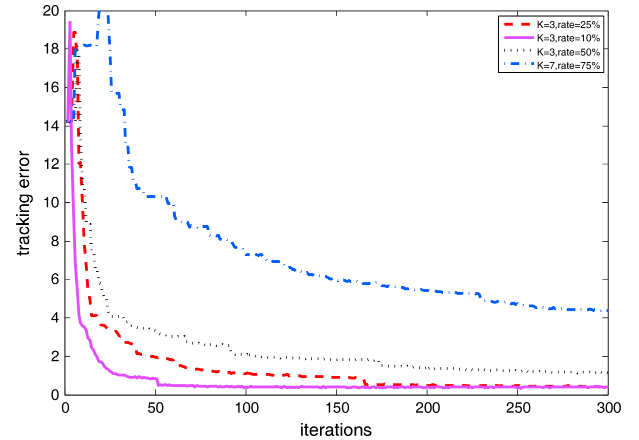

Fig. 7. The average absolute tracking error $\sqrt{\frac{\sum_{t=1}^{T}\|e_k(t)\|^2}{T}}$ for four different cases of K and loss rate.

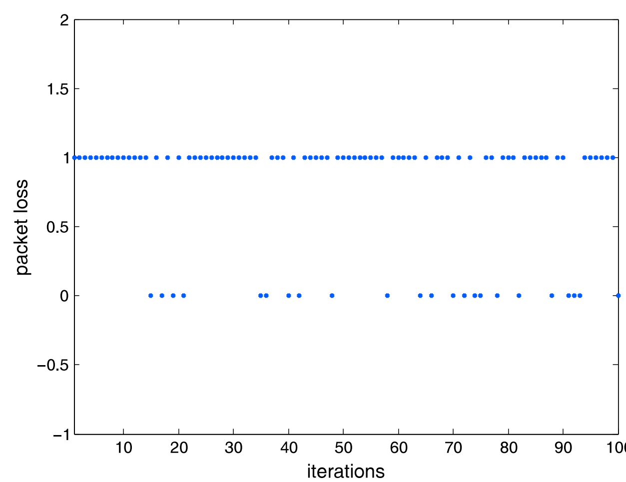

Fig. 8. Illustration of packet loss in the Bernoulli case for an arbitrary time instance.

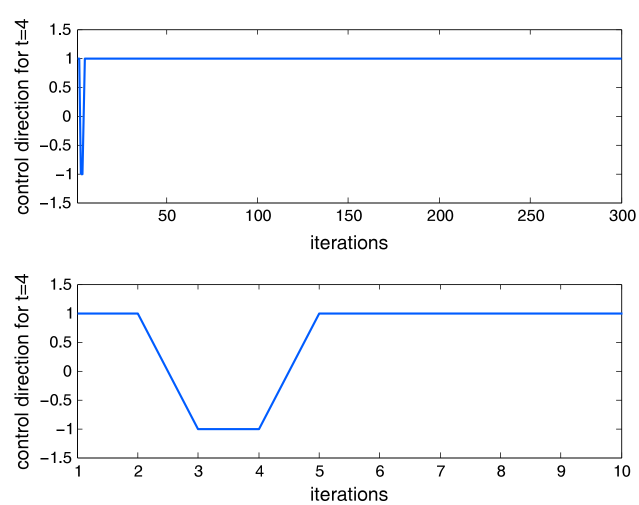

Fig. 9. Control direction regulation at $t = 4$ in the Bernoulli case.

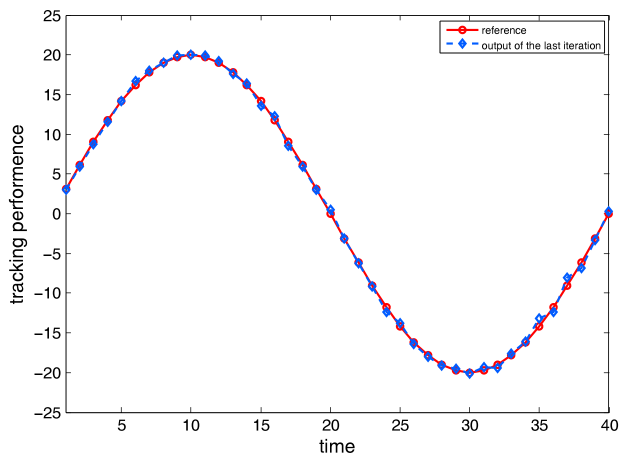

Fig. 10. $y_{300}(t)$ vs. $y_d(t)$ in the Bernoulli case.

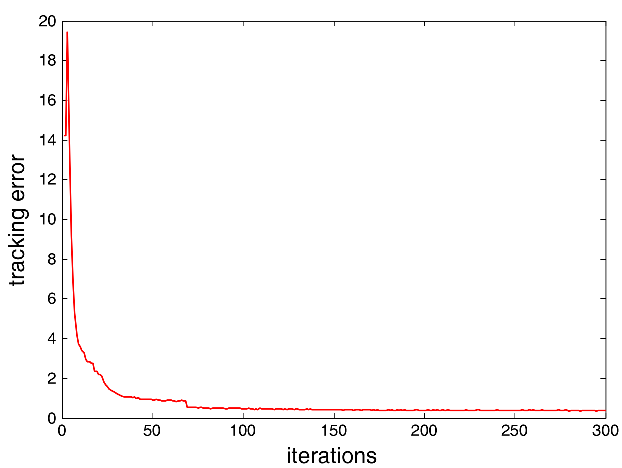

Fig. 11. The average absolute tracking error $\sqrt{\frac{\sum_{t=1}^{T}\|e_k(t)\|^2}{T}}$ in the Bernoulli case.

At the end of this section, some remarks are listed as follows. The rate of packet loss has great impact on the tracking performance. To be specific, the lower the loss rate is, the better the tracking performance behaves, and the sooner the convergence speed is. If the rate is zero, which means no packet is lost, then the update algorithm becomes a general iterative learning control algorithm. If the rate is 100%, which means no tracking information is fed back, then the update algorithm does not work anymore. For saving space, the simulation verification is omitted.

5. Conclusions

The ILC is considered for networked nonlinear systems with random measurement losses and unknown control direction. In this paper the random packet loss is modeled by an arbitrary stochastic sequence with bounded length requirement, which is different from many previous publications where the random packet loss is modeled by a binary Bernoulli random variable. Besides, the control direction, which plays an important role in the control design, is assumed unknown and a novel direction regulating approach is introduced in this paper. Based on this direction regulating approach, the P-type control update algorithm is proposed for SISO affine nonlinear system with stochastic measurement noises and the convergence with probability one is strictly proved. The results of this paper could be extended to MIMO case by minor modifications. An illustrative example is given to verify the effectiveness of the algorithm. For further research, other random packet loss models such as Markov chain would be interesting. Moreover, it is only the measurement loss that is considered in this paper, thus it is worth taking control loss or both of them into account.

Appendix

Proof of Lemma 1. The proof of this lemma can be carried out by induction along the time axis t. By (1) and (3),

$$\begin{aligned}\delta x_k(t+1) &= f_d(t) - f_k(t) + \mathbf{b}_d(t)u_d(t) - \mathbf{b}_k(t)u_k(t)\\ &= \delta f_k(t) + \delta\mathbf{b}_k(t)u_d(t) + \mathbf{b}_k(t)\delta u_k(t).\end{aligned} \tag{11}$$

Thus for $t = 0$, noticing A2 and A4, one has

$$\delta f_k(0) = f_d(0) - f_k(0) \xrightarrow[k\to\infty]{} 0$$

$$\delta\mathbf{b}_k(0) = \mathbf{b}_d(0) - \mathbf{b}_k(0) \xrightarrow[k\to\infty]{} 0$$

which imply that the first two terms at the right-hand of (11) tend to zero as $k \to \infty$. Since

$$\|\mathbf{b}_k(0)\| \le \|\mathbf{b}_d(0)\| + \|\delta\mathbf{b}_k(0)\|$$

it follows that $\mathbf{b}_k(0)$ is bounded. Thus if $\delta u_k(0) \xrightarrow[k\to\infty]{} 0$, then the third term at the right-hand of (11) also tends to zero. It further implies that $\delta x_k(1) \xrightarrow[k\to\infty]{} 0$ and then by A2 again, $\delta f_k(1) \xrightarrow[k\to\infty]{} 0$ and $\delta\mathbf{b}_k(1) \xrightarrow[k\to\infty]{} 0$. That is, the conclusion is valid for $t = 0$.

Now assume the conclusions of the lemma are true for $s = 0, 1, \ldots, t-1$, it suffices to show that the conclusions hold for t, i.e., $\|\delta x_k(t+1)\| \xrightarrow[k\to\infty]{} 0$, $\|\delta f_k(t+1)\| \xrightarrow[k\to\infty]{} 0$, $\|\delta\mathbf{b}_k(t+1)\| \xrightarrow[k\to\infty]{} 0$. This could be done the same argument as used above. This completes the proof. □

Proof of Lemma 2. If no truncation is considered in (4)–(5) for $t = 0$, it reduces to (7), which further leads to

$$\begin{aligned}\delta u_{k+1}(0) &= (1 - a_k I_{\{1\notin\mathcal{M}_k\}}\mathbf{c}^+\mathbf{b}_k(0))\delta u_k(0) - a_k I_{\{1\notin\mathcal{M}_k\}}\varphi_k(0)\\ &\quad + a_k I_{\{1\notin\mathcal{M}_k\}} w_k(1).\end{aligned} \tag{12}$$

Since $\mathbf{b}_k(0)$ is continuous in the initial state by A2, one has $\mathbf{b}_k(0) \xrightarrow[k\to\infty]{} \mathbf{b}_d(0)$ by A4 and $\mathbf{c}^+\mathbf{b}_k(0)$ converges to a positive constant by A3. Therefore by A6 it follows

$$\sum_{k=i}^{i+K-1}\left(-I_{\{1\notin\mathcal{M}_k\}}\mathbf{c}^+\mathbf{b}_k(0)\right) < -\gamma, \quad \gamma > 0 \tag{13}$$

for all sufficient large i.

Set $\phi_{i,j} \triangleq (1 - a_i I_{\{1\notin\mathcal{M}_i\}}\mathbf{c}^+\mathbf{b}_i(0))\cdots(1 - a_j I_{\{1\notin\mathcal{M}_j\}}\mathbf{c}^+\mathbf{b}_j(0))$, $i \ge j$, $\phi_{i,i+1} \triangleq 1$. It is clear that $1 - a_j I_{\{1\notin\mathcal{M}_j\}}\mathbf{c}^+\mathbf{b}_j(0) > 0$ for all large enough j, say, $j \ge j_0$. Then for any $i \ge j+K$, $j \ge j_0$ by (12) and (13), one has that

$$\begin{aligned}\phi_{i,j} &= \phi_{i-K,j}\left(1 - a_i\sum_{k=i-K+1}^{i} I_{\{1\notin\mathcal{M}_k\}}\mathbf{c}^+\mathbf{b}_k(0) + o(a_i)\right)\\ &\le \phi_{i-K,j}(1 - \gamma a_i + o(a_i))\\ &\le \phi_{i-K,j}\left(1 - \frac{\gamma}{K}\sum_{k=i-K+1}^{i} a_k + o(a_i)\right)\\ &\le \exp\left(-c\sum_{k=i-K+1}^{i} a_k\right)\phi_{i-K,j} \quad \text{with } c > 0.\end{aligned}$$

It follows from here that $\phi_{i,j} \le c_1\exp\left(-\frac{c}{2}\sum_{k=j}^{i} a_k\right)$, $\forall j \ge j_0$ for some $c_1 > 0$, and hence there is a c_2 such that

$$|\phi_{i,j}| \le c_2\exp\left(-\frac{c}{2}\sum_{k=j}^{i} a_k\right), \quad \forall i \ge j+K, j \ge j_0.$$

Therefore, for $\forall i \geq j_0 + K,\ \forall j \geq 0$ one has

$$|\phi_{i,j}| \leq |\phi_{i,j_0}|\,|\phi_{j_0-1,j}| \leq c_3 \exp\left(-\frac{c}{2}\sum_{k=j}^{i} a_k\right) \tag{14}$$

for some $c_3 > 0$.

From (12) it follows that

$$\delta u_{k+1}(0) = \phi_{k,0}\delta u_0(0) - \sum_{j=0}^{k}\phi_{k,j+1}a_j I_{\{1\notin\mathcal{M}_j\}}\varphi_j(0) + \sum_{j=0}^{k}\phi_{k,j+1}a_j I_{\{1\notin\mathcal{M}_j\}}w_j(1) \tag{15}$$

where the first term at the right-hand of the above equation tends to zero as $k \to \infty$ because of (14). By A2 and A4 it is clear that $\varphi_k(0) \xrightarrow[k\to\infty]{} 0$. By A5 it follows that

$$\sum_{k=1}^{\infty} a_k I_{\{1\notin\mathcal{M}_k\}}w_k(1) < 0.$$

Thus the last two terms at the right-hand side of (15) also tend to zero as $k \to \infty$ by using similar proof steps to Lemma 3.1.1 of [18]. Thus it concludes that $\delta u_k(0) \to 0$. In other words, the convergence of control sequence $u_k(0)$, $u_k(0) \xrightarrow[k\to\infty]{} u_d(0)$, has been proved. □

Proof of Lemma 3. If no truncation is considered in (4)–(5) for $t = 0$ and a_k is replaced by a_{k+m}, $\forall m$, it leads to

$$\begin{aligned}\delta u_{k+1}(0) &= \delta u_k(0) - a_{k+m}I_{\{1\notin\mathcal{M}_k\}}\mathbf{c}^+\mathbf{b}_k(0)\delta u_k(0)\\ &\quad - a_{k+m}I_{\{1\notin\mathcal{M}_k\}}[\varphi_k(0) - w_k(1)].\end{aligned}$$

Set

$$h_k = a_{k+m}$$

$$\psi_{i,j} = (1 - h_i I_{\{1\notin\mathcal{M}_i\}}\mathbf{c}^+\mathbf{b}_i(0))\cdots(1 - h_j I_{\{1\notin\mathcal{M}_j\}}\mathbf{c}^+\mathbf{b}_j(0)).$$

Then one has

$$\delta u_{k+1} = \psi_{k,0}\delta u_0(0) - \sum_{j=0}^{k}\psi_{k,j+1}h_j I_{\{1\notin\mathcal{M}_j\}}\phi_j(0) + \sum_{j=0}^{k}\psi_{k,j+1}h_j I_{\{1\notin\mathcal{M}_j\}}w_j(1). \tag{16}$$

To prove the uniform boundedness of $\delta u_k(0)$ for any $m > 0$ and fixed initial value $\delta u_0(0)$, it suffices to show that all the three terms at the right-hand side of (16) are uniformly bounded with respect to m.

Similar to the proof of Lemma 2, one has

$$|\psi_{i,j}| \leq c_4 \exp\left(-c\sum_{k=j}^{i} h_k\right), \quad \forall i \geq j,\ \forall j \geq 0 \tag{17}$$

for some constants $c_4 > 0$ and $c > 0$.

Then it is obvious $|\psi_{k,0}| \leq c_4,\ \forall m > 0$, and hence

$$|\psi_{k,0}\delta u_0(0)| \leq c_4|\delta u_0(0)|. \tag{18}$$

This shows the uniform boundedness of the first term at the right-hand side of (16).

Since $\varphi_k(0) \xrightarrow[k\to\infty]{} 0$, $\sup_k|\varphi_k(0)| < \infty$. By noticing that $h_k \to 0$, there is k_1 such that

$$h_k \leq 2\left(h_k - \frac{ch_k^2}{2}\right), \quad \text{and} \quad 0 < ch_k < 1, \quad \forall m > 0$$

$\forall k \geq k_1$. Hence, it follows that

$$\begin{aligned}&\left|\sum_{j=k_1}^{k}\psi_{k,j+1}h_j I_{\{1\notin\mathcal{M}_j\}}\varphi_j(0)\right|\\
&\leq c_4\sup_k|\varphi_k(0)|\sum_{j=k_1}^{k}\left[\exp\left(-c\sum_{i=j+1}^{k}h_i\right)h_j\right]\\
&\leq 2c_4\sup_k|\varphi_k(0)|\sum_{j=k_1}^{k}\left(h_j - \frac{ch_j^2}{2}\right)\exp\left(-c\sum_{i=j+1}^{k}h_i\right)\\
&\leq \frac{2c_4}{c}\sup_k|\varphi_k(0)|\sum_{j=k_1}^{k}(1-e^{-ch_j})\exp\left(-c\sum_{i=j+1}^{k}h_i\right)\\
&= \frac{2c_4}{c}\sup_k|\varphi_k(0)|\sum_{j=k_1}^{k}\left[\exp\left(-c\sum_{i=j+1}^{k}h_i\right) - \exp\left(-c\sum_{i=j}^{k}h_i\right)\right]\\
&\leq \frac{2c_4}{c}\sup_k|\varphi_k(0)|.\end{aligned}$$

Besides, it is clear that $|\sum_{j=0}^{k_1}\psi_{k,j+1}h_j I_{\{1\notin\mathcal{M}_j\}}\varphi_j(0)|$ is bounded. Thus the uniform boundedness for the second term at the right-hand side of (16) is proved.

Now check the last term $\sum_{j=0}^{k}\psi_{k,j+1}h_j I_{\{1\notin\mathcal{M}_j\}}w_j(1)$. By the definition of h_k, it is clear $\sum_{k=1}^{\infty}h_k = \infty$ and $\sum_{k=1}^{\infty}h_k^2 < \infty$. Let

$$s_k = \sum_{j=1}^{k}h_j I_{\{1\notin\mathcal{M}_j\}}w_j(1), \quad s_{-1} = 0. \tag{19}$$

By A5 it is clear $s_k \to s < \infty$. Hence, for any $\epsilon > 0$, there is k_2 such that $\|s_j - s\| \leq \epsilon,\ \forall j \geq k_2$. By a partial summation

$$\begin{aligned}&\sum_{j=0}^{k}\psi_{k,j+1}h_j I_{\{1\notin\mathcal{M}_j\}}w_j(1) = \sum_{j=0}^{k}\psi_{k,j+1}(s_j - s_{j-1})\\
&= s_k - \sum_{j=0}^{k}(\psi_{k,j+1} - \psi_{k,j})s_{j-1}\\
&= s_k - \sum_{j=0}^{k}(\psi_{k,j+1} - \psi_{k,j})s - \sum_{j=0}^{k}(\psi_{k,j+1} - \psi_{k,j})(s_{j-1} - s)\\
&= s_k - s + \psi_{k,0}s - \sum_{j=0}^{k_2}(\psi_{k,j+1} - \psi_{k,j})(s_{j-1} - s)\\
&\quad - \sum_{j=k_2}^{k}\psi_{k,j+1}h_j I_{\{1\notin\mathcal{M}_j\}}\mathbf{c}^+\mathbf{b}_k(0)(s_{j-1} - s)\end{aligned}$$

where all terms tend to zero as $k \to \infty$ except the last one. By noticing $\|s_j - s\| \leq \epsilon,\ \forall j \geq k_2$, one has

$$\begin{aligned}&\left\|\sum_{j=k_2}^{k}\psi_{k,j+1}h_j I_{\{1\notin\mathcal{M}_j\}}\mathbf{c}^+\mathbf{b}_k(0)(s_{j-1} - s)\right\|\\
&\leq \epsilon\sup_k|\mathbf{c}^+\mathbf{b}_k(0)|\cdot\left\|\sum_{j=k_2}^{k}\psi_{k,j+1}h_j\right\|\end{aligned}$$

which tends to zero as $k \to \infty$ and $\epsilon \to 0$. Thus the third term at the right-hand side of (16) is also uniformly bounded. The proof of this lemma is completed. □

Proof of Lemma 4. It is observed from Lemma 2, Lemma 3 and first part proof of Theorem 2 that the algorithm (4)–(6) may have only finite number of truncations for $t = 0$.

If the algorithm stops at the correct direction, then by Lemma 2 the algorithm converges to the optimal control.

For the impossibility of stopping at the wrong direction, one may consider the following regression

$$u_{k+1}(0) = u_k(0) - a_k I_{\{1\notin\mathcal{M}_k\}}\mathbf{c}^+\mathbf{b}_k(0)\delta u_k(0) - a_k I_{\{1\notin\mathcal{M}_k\}}\varphi_k(0) + a_k I_{\{1\notin\mathcal{M}_k\}} w_k(1). \tag{20}$$

The proof consists of two steps. The first is to show that $u_k(0)$ defined by (20) converges and the second is to show that $u_k(0)$ cannot converge to a point $u'(0) \neq u_d(0)$.

First, assume that $u_k(0)$ defined by (20) is not convergent. Take a Lyapunov function $\zeta(x) \triangleq (u_d(0) - x)^2$, then one has

$$0 \le \liminf_{k\to\infty} \zeta(u_k(0)) < \limsup_{k\to\infty} \zeta(u_k(0)) < \infty. \tag{21}$$

We say that $\zeta(x_{n_k}), \dots, \zeta(x_{m_k})$ down-cross the interval $[\delta_1, \delta_2]$, if $\zeta(x_{n_k}) \ge \delta_2$, $\zeta(x_{m_k}) \le \delta_1$, and $\delta_1 < \zeta(x_j) < \delta_2$, $\forall j : n_k < j < m_k$. From (21) it follows that $\zeta(u_k(0))$ down-cross a nonempty interval $[\delta_1, \delta_2]$ infinitely many times. Without loss of generality, it is assumed $d([\delta_1, \delta_2], \zeta(u_d(0))) > 0$, where $d(\cdot,\cdot)$ denotes the distance.

Let $\zeta(u_{n_k}(0)), \dots, \zeta(u_{m_k}(0))$, $k = 1, 2, \dots$, be down-crossings.

By the boundedness of $u_k(0)$, without loss of generality, one may assume $u_{n_k}(0) \xrightarrow[k\to\infty]{} \bar{u}$.

Set $m(k, T) \triangleq \max\{m : \sum_{i=k}^{m} a_i I_{\{1\notin\mathcal{M}_i\}} \le T\}$. For sufficiently large k and small enough T one has

$$\begin{aligned}|u_{m+1}(0) - u_{n_k}(0)| &\le \left|\sum_{j=n_k}^{m} a_j I_{\{1\notin\mathcal{M}_j\}}\mathbf{c}^+\mathbf{b}_j(0)\delta u_j(0)\right| \\ &\quad + \left|\sum_{j=n_k}^{m} a_j I_{\{1\notin\mathcal{M}_j\}}\mathbf{c}^+\mathbf{b}_j(0)\varphi_j(0)\right| + \left|\sum_{j=n_k}^{m} a_j I_{\{1\notin\mathcal{M}_j\}}\mathbf{c}^+\mathbf{b}_j(0) w_j(1)\right| \\ &\le \sup_j |\mathbf{c}^+\mathbf{b}_j(0)\delta u_j(0)| \sum_{j=n_k}^{m} a_j I_{\{1\notin\mathcal{M}_j\}} + o(\tau) \\ &\le c_5 r \quad \forall m : n_k \le m \le m(n_k, \tau),\ \forall \tau \in [0, T]\end{aligned} \tag{22}$$

where c_5 is a positive constant and $o(\cdot)$ denotes infinitesimal of high order.

Set $\tau = a_{n_k} I_{\{1\notin\mathcal{M}_{n_k}\}}$ in the above expression, one has

$$|u_{n_k+1}(0) - u_{n_k}(0)| \le c_5 a_{n_k} I_{\{1\notin\mathcal{M}_{n_k}\}} \xrightarrow[k\to\infty]{} 0. \tag{23}$$

By the definition of down-crossing $\zeta(u_{n_k}) \ge \delta_2 > \zeta(u_{n_k+1})$ it follows that

$$\zeta(u_{n_k}(0)) \xrightarrow[k\to\infty]{} \delta_2 = \zeta(\bar{u}), \qquad d(\bar{u}, u_d(0)) \triangleq \delta > 0. \tag{24}$$

For small enough τ and sufficiently large k one has

$$d(u_m(0), u_d(0)) \ge \frac{\delta}{2}, \quad \forall m : n_k \le m \le m(n_k, \tau). \tag{25}$$

Consequently for large enough k it follows

$$\begin{aligned}&\zeta(u_{m(n_k,\tau)+1}(0)) - \zeta(u_{n_k}(0)) \\ &= -\sum_{j=n_k}^{m(n_k,\tau)} a_j I_{\{1\notin\mathcal{M}_j\}}(y_d(1) - y_j(1))\zeta_u(\bar{u}) + o(\tau) \\ &= -\sum_{j=n_k}^{m(n_k,\tau)} a_j I_{\{1\notin\mathcal{M}_j\}}\mathbf{c}^+\mathbf{b}_j(0)\delta u_j(0)\zeta_u(u_j(0)) \\ &\quad + \sum_{j=n_k}^{m(n_k,\tau)} a_j I_{\{1\notin\mathcal{M}_j\}}\mathbf{c}^+\mathbf{b}_j(0)\delta u_j(0)(\zeta_u(u_j(0)) - \zeta_u(\bar{u})) \\ &\quad - \sum_{j=n_k}^{m(n_k,\tau)} a_j I_{\{1\notin\mathcal{M}_j\}}[\varphi_j(0) - w_j(1)]\zeta_u(\bar{u}) + o(\tau).\end{aligned}$$

Since $\varphi_k(0) \to 0$ one has that

$$\limsup_{k\to\infty}\left|\zeta_u(\bar{u})\sum_{j=n_k}^{m(n_k,\tau)} a_j I_{\{1\notin\mathcal{M}_j\}}[\varphi_j(0) - w_j(1)]\right| = o(\tau) \tag{26}$$

and

$$\left|\sum_{j=n_k}^{m(n_k,\tau)} a_j I_{\{1\notin\mathcal{M}_j\}}\mathbf{c}^+\mathbf{b}_j(0)\delta u_j(0)(\zeta_u(u_j(0)) - \zeta_u(\bar{u}))\right| = o(\tau). \tag{27}$$

By (25) and A3 there exists an $\alpha > 0$ such that

$$-\mathbf{c}^+\mathbf{b}_j(0)\delta u_j(0)\zeta_u(u_j(0)) = 2\mathbf{c}^+\mathbf{b}_j(0)[\delta u_j(0)]^2 > \alpha, \quad \forall j : n_k \le j \le m(n_k, \tau). \tag{28}$$

Combining the last four formulas leads to

$$\zeta(u_{m(n_k,\tau)+1}(0)) - \zeta(u_{n_k}(0)) \ge \beta\tau \tag{29}$$

for some $\beta > 0$ and $\tau > 0$, if k is sufficiently large.

From (24) and (29) it follows that

$$\liminf_{k\to\infty} \zeta(u_{m(n_k,\tau)+1}(0)) \ge \delta_2 + \beta\tau. \tag{30}$$

On the other hand, from (22) it is observed that

$$\lim_{\tau\to 0} \max_{n_k \le j \le m(n_k,\tau)} |\zeta(u_{j+1}(0)) - \zeta(u_{n_k}(0))| = 0. \tag{31}$$

This means that $m(n_k, \tau) + 1 < m_k$ for small enough τ, and hence $\zeta(u_{m(n_k,\tau)+1}(0)) \in (\delta_1, \delta_2]$. This contradicts with (30). Hence, $\{\zeta(u_k(0))\}$ is impossible to have infinitely many down-crossings, and $\{u_k(0)\}$ is convergent.

Then it comes to show that $\{u_k(0)\}$ cannot converge to a point $u'(0) \neq u_d(0)$. To show this, assume the converse: $u_k(0) \xrightarrow[k\to\infty]{} u'(0) \neq u_d(0)$.

From (20) it follows

$$u_{k_0+n}(0) = u_{k_0}(0) - \sum_{j=k_0}^{k_0+n-1} a_j I_{\{1\notin\mathcal{M}_j\}}[\mathbf{c}^+\mathbf{b}_j(0)\delta u_j(0) + \varphi_j(0)] + \sum_{j=k_0}^{k_0+n-1} a_j I_{\{1\notin\mathcal{M}_j\}} w_j(1), \quad \forall n. \tag{32}$$

It is clear that

$$\sum_{j=k_0}^{\infty} a_j I_{\{1\notin\mathcal{M}_j\}} w_j(1) < \infty \tag{33}$$

and by A4, A2, $\varphi_j(0) \xrightarrow[j\to\infty]{} 0$. Since $u'(0) \neq u_d(0)$, $\mathbf{c}^+\mathbf{b}_j(0)\delta u_j(0)$ would converge to a nonzero constant. Then one has

$$\sum_{j=k_0}^{k_0+n-1} a_j I_{\{1\notin\mathcal{M}_j\}}[\mathbf{c}^+\mathbf{b}_j(0)\delta u_j(0) + \varphi_j(0)] \xrightarrow[n\to\infty]{} \infty \tag{34}$$

which implies $u_{k_0+n}(0) \xrightarrow[n\to\infty]{} \infty$. This, however, contradicts with the boundedness of $\{u_k(0)\}$. Thus the proof is completed. □

References

[1] D.A. Bristow, M. Tharayil, A.G. Alleyne, A survey of iterative learning control: a learning-based method for high-performance tracking control, IEEE Control Syst. Mag. 26 (3) (2006) 96–114.

[2] H.-S. Ahn, Y.Q. Chen, K.L. Moore, Iterative learning control: survey and categorization from 1998 to 2004, IEEE Trans. Syst. Man. Cybern. C 37 (6) (2007) 1099–1121.

[3] Y. Wang, F. Gao, F.J. Doyle III, Survey on iterative learning control, repetitive control, and run-to-run control, J. Process Control 19 (10) (2009) 1589–1600.

[4] D. Shen, Y. Wang, Survey on stochastic iterative learning control, J. Process Control 24 (12) (2014) 64–77.

[5] S.S. Saab, A discrete-time stochastic learning control algorithm, IEEE Trans. Automat. Control 46 (6) (2001) 877–887.
[6] H.S. Ahn, Y.Q. Chen, K.L. Moore, Intermittent iterative learning control, in: Proc. the 2006 IEEE Int. Symposium on Intelligent Control, 2006, pp. 832–837.
[7] H.S. Ahn, K.L. Moore, Y.Q. Chen, Discrete-time intermittent iterative learning controller with independent data dropouts, in: Proc. the 2008 IFAC World Congress, 2008, pp. 12442–12447.
[8] H.S. Ahn, K.L. Moore, Y.Q. Chen, Stability of discrete-time iterative learning control with random data dropouts and delayed controlled signals in networked control systems, in: Proc. the IEEE Int. Conf. Control Automation, Robotics, and Vision, 2008.
[9] X. Bu, Z.-S. Hou, Stability of iterative learning control with data dropouts via asynchronous dynamical system, Int. J. Autom. Comput. 8 (1) (2011) 29–36.
[10] X. Bu, Z.-S. Hou, F. Yu, Stability of first and high order iterative learning control with data dropouts, Int. J. Control Autom. Syst. 9 (5) (2011) 843–849.
[11] X. Bu, F. Yu, Z.-S. Hou, F. Wang, Iterative learning control for a class of nonlinear systems with measurement dropouts, Control Theory Appl. 29 (11) (2012) 1458–1464.
[12] X. Bu, F. Yu, Z.-S. Hou, F. Wang, Iterative learning control for a class of nonlinear systems with random packet losses, Nonlinear Anal. RWA 14 (1) (2013) 567–580.
[13] A. Hassibi, S.P. Boyd, J.P. How, Control of asynchronous dynamical systems with rate constraints on events, in: Proc. The 38th IEEE Conf. Decision and Control, 1999, pp. 1345–1351.
[14] R.D. Nussbaum, Some remarks on the conjecture in parameter adaptive control, Systems Control Lett. 3 (1983) 243–246.
[15] H. Chen, P. Jiang, Adaptive iterative learning control for nonlinear systems with unknown control gain, ASME J. Dyn. Syst. Meas. Control 126 (2004) 916–920.
[16] J.-X. Xu, R. Yan, Iterative learning control design without a prior knowledge of the control direction, Automatica 40 (10) (2004) 1803–1809.
[17] H.F. Chen, Almost surely convergence of iterative learning control for stochastic systems, Sci. China Ser. F 46 (1) (2003) 69–79.
[18] H.F. Chen, Stochastic Approximation and its Applications, Kluwer, Dordrecht, The Netherlands, 2002.
[19] D. Shen, Z.S. Hou, Iterative learning control with unknown control direction: a novel data-based approach, IEEE Trans. Neural Netw. 22 (12) (2011) 2237–2249.
[20] P. Jiang, H. Chen, C.A. Bamforth, A universal iterative learning stabilizer for a class of MIMO systems, Automatica 42 (6) (2006) 973–981.
[21] J.-X. Xu, R. Yan, On initial conditions in iterative learning control, IEEE Trans. Automat. Control 50 (9) (2005) 1349–1354.
[22] M. Sun, D. Wang, Initial shift issues on discrete-time iterative learning control with system relative degree, IEEE Trans. Automat. Control 48 (1) (2003) 144–148.
[23] Y.Q. Chen, C. Wen, Z. Gong, M. Sun, An iterative learning controller with initial state learning, IEEE Trans. Automat. Control 44 (2) (1999) 371–376.
[24] D. Shen, Iterative learning control for nonlinear stochastic systems (Ph.D. thesis), Chinese Academy of Sciences, 2010.