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# Iterative learning control for networked stochastic systems with random packet losses

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The iterative learning control is constructed for discrete-time stochastic systems with random measurement losses modelled by a stochastic sequence. A simple P-type update law is used and the almost sure convergence is strictly proved for both linear case and nonlinear case based on stochastic approximation. Illustrative examples show the effectiveness of the proposed approach.

**Keywords:** iterative learning control; networked stochastic system; random packet losses; stochastic approximation



## 1. Introduction

In many practical applications such as chemical process, hard disk drives, and robotics, the system would perform the same task repeatedly. For this case, the information generated in the previous batches would help us to improve the performance if we use the information in the design for the next batch. This intuitive idea motivates the developments of iterative learning control (ILC), which has become one important intelligent control strategy as it could realise accurate tracking with simple control algorithm (Ahn, Chen, & Moore, 2007; Cai, Chen, Angland, & Zhang, 2014; Nygren, Pelckmans, & Carlsson, 2014; Owens, Freeman, & Chu, 2013; Wang, Gao, & Doyle III, 2009). Moreover, as we could see, stochastic noises are inevitable in many actual systems because of complicated environment factors and random disturbances, thus the discussions on stochastic ILC are published recently (Shen & Wang, 2014). Unlike deterministic systems, the stochastic case would depend more on probabilistic framework since the traditional design and analysis methods such as contraction mapping method are no longer suitable. Furthermore, networked control systems are widely applied as they own much more flexibility to deal with complex systems. However, in networked control systems, the data packets can be lost during transmission in consequence of network congestion, linkage interrupt, transmission error, and etc., which further could reduce the system performance. This motivates the research on ILC for stochastic systems with random packet losses.

Although random packet losses have been discussed numerously in conventional networked control systems, the publications related to ILC are very rare. We have tried our best to search the literature, but the outcomes are limited, which is a side-reflection that this topic is on the initial step. In most papers, the packet loss is modelled as a Bernoulli random variable, whose value is 1 when the packet is successfully transmitted and 0 otherwise. Bu and co-workers considered ILC for networked control systems from the statistics point of view (Bu & Hou, 2011; Bu, Hou, & Yu, 2011; Bu, Hou, Yu, & Fu, 2014; Bu, Yu, Hou, & Wang, 2012, 2013). In Bu and Hou (2011), the linear time-invariant discrete system was lifted into the so-called super-vector form, consequently the model turned into 1-D form which only evolved along the iteration axis. Then the iteration equation of tracking errors was directly obtained. Based on the equation and exponential stability for asynchronous dynamical systems (Hassibi, Boyd, & How, 1999), the stability analysis was given in Bu and Hou (2011). Another stability result was given in Bu et al. (2011) for single-input-single-output (SISO) linear time-invariant system under packet losses. Unlike Bu and Hou (2011), the latter took mathematical expectations to both sides of the iteration equation of tracking errors directly, and then gave the stability condition according to the mathematical expectation of tracking error. Further results were given in Bu et al. (2014) where the relationship between data loss rate and convergence speed was proposed according to the expectations of tracking errors. Both Bu et al. (2012) and Bu et al. (2013) considered the nonlinear system case. However, because of the existence of nonlinearity, the conventional lifting technique was no longer applicable here. Instead, the regression equation was first established, and then regression inequality was given by taking expectations and contract mapping method, whence the effect of stochastic packet loss was eliminated so that the convergence condition was provided. In short, the main idea of Bu's series publications is transforming the stochastic equation/equality into deterministic

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equation/equality by taking mathematical expectations and then showing the stability/convergence conditions.

Ahn and co-workers also considered this topic for multiple-input-multiple-output (MIMO) time-invariant systems and showed the mean-square stability under packet losses by the Kalman filtering-based techniques for ILC (Ahn, Chen, & Moore, 2006; Ahn, Moore, & Chen, 2008a, b). The major differences among these papers are the packet loss locations. In particular, in Ahn et al. (2006) only the measurement output was assumed to be randomly lost when transmitting back to control centre. Besides, the output vector was assumed to be completely lost if packet loss occurs. In practical system, maybe only part of the multi-dimensional output is lost but the other part is transmitted back. The analysis of this case was considered in Ahn et al. (2008a). Ahn et al. (2008b) further discussed the case that packet loss happened to the control signals as well as output signals. The Kalman filtering techniques proposed in Saab (2001) was applied, thus all the convergence results are in mean-square sense.

In addition, Liu, Xu, and Wu (2009) considered linear time-invariant system with repetitive disturbances and reduced the effect of packet losses by introducing an average-operator along iteration axis. On the basis of traditional contraction mapping method, it was shown that the mathematical expectation of the averaged tracking error would converge to a bounded zone. Roughly speaking, with the help of averaged historical information, better tracking performance is hopeful, which requires further studies.

To sum up, this research on ILC for networked control system with random packet loss focuses the following three aspects: the convergence is in mean-square sense or mathematical expectation sense; the random packet loss is modelled by a binary Bernoulli random variable; and none stochastic noise is involved except Ahn et al. (2008a). In this paper, the random packet loss is modelled as an arbitrary stochastic sequence with suitable conditions, which differs from the current formulations. A P-type ILC algorithm is proposed for SISO time-varying stochastic system and proved to be almost sure convergent based on stochastic approximation algorithm. The major differences between this study and Bu's and Ahn's papers are twofold: the stochastic sequence formulation of packet loss which could not be handled by Bu's or Ahn's methods, and the almost sure convergence under stochastic noises and packet losses.

Besides, the stochastic approximation technique used in this paper is a little different to the original one. In the original stochastic approximation algorithm, the estimation updates at every step, while in this paper the algorithm may not update at some steps. This is because there is packet losses in our problem, and the control input does not update if the corresponding packet loses since the tracking performance information is lost. The stochastic approximation technique has been used in Shen and Chen (2012) for large-scale systems. However, there are some crucial differences between that paper and this paper. First of all, the asynchronisation problem is considered in Shen and Chen (2012) while packet losses problem is dealt within this paper. Next, the output of the whole iteration is taken into account in Shen and Chen (2012), in contrast to which, the packet loss is modelled according to each time instance independently. Moreover, the detailed design of control algorithm is different and consequently the proofs are also disparate.

The rest of the paper are arranged as follows: Section 2 provides the system formulation and the definition of packet loss, as well as a preliminary lemma; Section 3 gives the ILC algorithm and the almost sure convergence analysis; Section 4 extends the linear system to the affine nonlinear case and the convergence result; Section 5 provides illustrative examples to show the effectiveness; some concluding remarks are given in Section 6.

## 2. Problem formulation

Consider the following SISO time-varying linear discrete system

$$x_k(t+1) = A(t)x_k(t) + \mathbf{b}(t)u_k(t) + w_k(t+1)$$
$$y_k(t) = \mathbf{c}(t)x_k(t) + v_k(t), \quad (1)$$

where $t \in \{0, 1, \dots, N\}$ denotes the time instance in an iteration of the process, while $k = 1, 2, \dots$ labels different iteration, $u_k(t) \in \mathbb{R}$, $x_k(t) \in \mathbb{R}^n$, and $y_k(t) \in \mathbb{R}$ denote the input, state, and output, respectively. $A(t)$, $\mathbf{b}(t)$, and $\mathbf{c}(t)$ denote unknown system information. $w_k(t)$ and $v_k(t)$ are stochastic system noise and measurement noise, respectively.

The setup of the control system is illustrated in Figure 1. For convenience, only measurement loss is con-

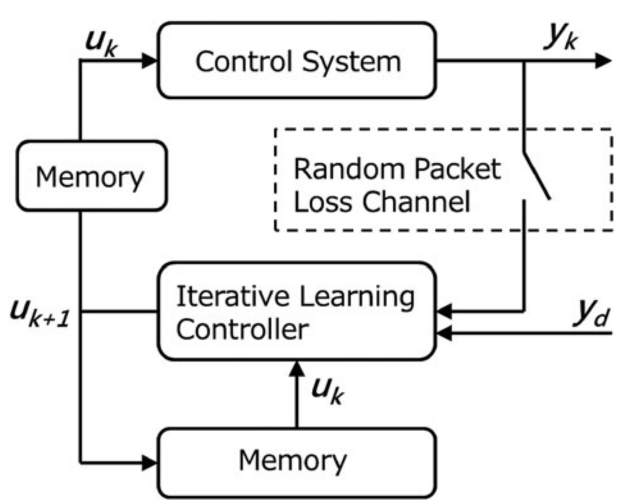


Figure 1. Block diagram of networked control system with measurement packet loss.

sidered in this short paper, which could be easily extended to the case that both measurement and control signals are lossy. As shown in Figure 1, the measurement signals are transmitted back through a lossy channel, which could be regarded as a switch that opens and closes in a random manner. In previous publications, the random packet loss is modelled as a binary Bernoulli random variable. However, in this paper, a random sequence model of the random packet loss is proposed. To this end, denote $\mathcal{M}_k$ as the random set of time locations at which measurements are lost in the $k$th iteration. In other words, $t_0 \in \mathcal{M}_k$ if $y_k(t_0)$ is lost.

Let $\mathcal{F}_k \triangleq \sigma\{y_j(t), x_j(t), w_j(t), v_j(t), 0 \le j \le k, t \in \{0, \ldots, N\}\}$ be the $\sigma$-algebra generated by $y_j(t), x_j(t), w_j(t), v_j(t)$, $0 \le t \le N$, $0 \le j \le k$. Then the set of admissible controls is defined as

$$U = \{u_{k+1}(t) \in \mathcal{F}_k, \sup_k \|u_k(t)\| < \infty, \text{a.s.}\ t \in \{0, \ldots, N-1\}, k = 0, 1, 2, \ldots\}.$$

The control purpose is to find the control sequence $\{u_k(t), k = 0, 1, 2, \ldots\} \subset U$ under random packet loss environment to minimise the following averaged tracking errors, $\forall t \in \{0, 1, \ldots, N\}$

$$V(t) = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \|y_k(t) - y_d(t)\|^2, \tag{2}$$

where $y_d(t)$, $t \in \{0, 1, \ldots, N\}$ is the tracking target. The following assumptions are used.

**Assumption 1:** *The real number* $\mathbf{c}(t+1)\mathbf{b}(t)$ *coupling the input and output is an unknown nonzero constant, but its sign, characterising the control direction, is assumed known.*

Without loss of generality, it is assumed that $\mathbf{c}(t+1)\mathbf{b}(t) > 0$ in the rest of the paper. It is noted that $\mathbf{c}(t+1)\mathbf{b}(t) \neq 0$ is equivalent to claiming that the input/output relative degree is one. In the following, without causing confusion, the symbol $\mathbf{c}(t+1)\mathbf{b}(t)$ will be abbreviated as $\mathbf{c}^+\mathbf{b}(t)$ for simplicity of writing.

Under Assumption 1 and the system (1), it is noted there exist suitable initial state value $x_d(0)$ and input $u_d(t)$ such that

$$\begin{aligned} x_d(t+1) &= A(t)x_d(t) + \mathbf{b}(t)u_d(t) \\ y_d(t) &= \mathbf{c}(t)x_d(t). \end{aligned} \tag{3}$$

By Assumption 1 and Equation (3), one has

$$u_d(t) = (\mathbf{c}^+\mathbf{b}(t))^{-1}(y_d(t+1) - \mathbf{c}(t+1)A(t)x_d(t)).$$

**Assumption 2:** *For each t, the measurement packet loss is random without obeying any certain probability distribution, but there is a number K such that during successive K iterations, at least in one iteration, the measurement is successfully sent back.*

The number $K$ is not necessary to be known prior, in other words, only the existence of such number is required. Thus, this condition means that the measurements should not be lost too much to guarantee the convergence in almost sure sense. It is worth pointing out that this model of packet loss is different from the traditional binary Bernoulli one, which could not be covered by Assumption 2 and vice versa. Besides, the new model is another suitable description of practical packet losses.

**Assumption 3:** *For each t, the independent and identically distributed (iid) sequence* $\{w_k(t), k = 0, 1, \ldots\}$ *is independent of the iid sequence* $\{v_k(t), k = 0, 1, \ldots\}$ *with* $\mathbb{E}w_k(t) = 0$, $\mathbb{E}v_k(t) = 0$, $\sup_k \mathbb{E}w_k^2(t) < \infty$, $\sup_k \mathbb{E}v_k^2(t) < \infty$, $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} w_k^2(t) = R_w^t$, *and* $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} v_k^2(t) = R_v^t$, *a.s., where* $R_w^t$ *and* $R_v^2$ *are unknown.*

Note that the noise assumption is made according to the iteration axis rather than the time axis, thus, this requirement is not rigorous as the process would be performed repeatedly.

**Assumption 4:** *The sequence of initial values* $\{x_k(0)\}$ *is iid with* $\mathbb{E}x_k(0) = x_d(0)$, $\sup_k \mathbb{E}x_k^2(0) < \infty$, *and* $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} x_k^2(0) = R_0$. *Furthermore, the sequences* $\{x_k(0), k = 0, 1, \ldots\}$, $\{w_k(t), k = 0, 1, \ldots\}$, *and* $\{v_k(t), k = 0, 1, \ldots\}$ *are mutually independent.*

To facilitate the expression, denote $w_k(0) = x_k(0) - x_d(0)$. Then, it is easy to define $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} w_k^2(0) = R_w^0$ to satisfy the formulation of Assumption 3. In order to give a control update algorithm, the optimal control which minimises Equation (2) should be first shown as in the following lemma.

**Lemma 1:** *For the stochastic system* (1) *and tracking objective* $y_d(t+1)$, *assume Assumptions 1, 3, and 4 hold, then for any arbitrary time* $t+1$, *the index* (2) *will be minimised if the control sequence* $\{u_k(t)\}$ *is admissible and satisfies* $u_k(i) \xrightarrow[k\to\infty]{} u_d(i)$, $i = 0, 1, \ldots, t$. *In this case,* $\{u_k(t)\}$ *is called the optimal control sequence.*

**Proof:** For simplicity of expression, denote $\delta x_k(t) \triangleq x_d(t) - x_k(t)$ and $\delta u_k(t) \triangleq u_d(t) - u_k(t)$. By Equations (1) and (3),

$$\delta x_k(t+1) = A(t)\delta x_k(t) + \mathbf{b}(t)\delta u_k(t) - w_k(t+1).$$

By backwardly iterating this equation, we have

$$\delta x_k(t+1) = \sum_{i=1}^{t+1}\left(\prod_{l=i}^{t} A(l)\right)\mathbf{b}(i-1)\delta u_k(i-1) - \sum_{i=0}^{t+1}\left(\prod_{l=i}^{t} A(l)\right) w_k(i),$$

where $\prod_{l=i}^{j} A(l) \triangleq A(j)A(j-1)\dots A(i),\ j \geq i$ and $\prod_{l=i}^{j} A(l) = I, j < i$. Thereby,

$$\begin{aligned} y_d(t+1) - y_k(t+1) &= \mathbf{c}(t+1)\delta x_k(t+1) - v_k(t+1) \\ &= \phi_k(t+1) + \varphi_k(t+1) - v_k(t+1), \end{aligned}$$

where

$$\phi_k(t+1) = \mathbf{c}(t+1)\sum_{i=1}^{t+1}\left(\prod_{l=i}^{t} A(l)\right)\mathbf{b}(i-1)\delta u_k(i-1)$$

$$\varphi_k(t+1) = \mathbf{c}(t+1)\sum_{i=0}^{t+1}\left(\prod_{l=i}^{t} A(l)\right) w_k(i).$$

By Assumptions 3 and 4 and noticing that $u_k(i) \in \mathcal{F}_{k-1}$, $i = 0, 1, \dots, t$, it is clear that $\phi_k(t+1)$, $\varphi_k(t+1)$, and $v_k(t+1)$ are mutually independent.

By using Theorem 2.8 of Chen and Guo (1991)

$$\begin{aligned} &\sum_{k=1}^{n} \phi_k(t+1)(\varphi_k(t+1) - v_k(t+1)) \\ &\quad = O\left(\left(\sum_{k=1}^{n} \|\phi_k(t+1)\|^2\right)^{\frac{1}{2}+\eta}\right), \quad a.s. \forall \eta > 0 \\ &\sum_{k=1}^{n} \varphi_k(t+1) v_k(t+1) \\ &\quad = O\left(\left(\sum_{k=1}^{n} \|v_k(t+1)\|^2\right)^{\frac{1}{2}+\eta}\right), \quad a.s. \forall \eta > 0. \end{aligned}$$

Consequently,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \|y_d(t+1) - y_k(t+1)\|^2 \\ &= \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \|\phi_k(t+1)\|^2 \\ &\quad + \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \|\varphi_k(t+1)\|^2 \\ &\quad + \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \|v_k(t+1)\|^2 \\ &= \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \|\phi_k(t+1)\|^2 \\ &\quad + \mathrm{tr}\left[\mathbf{c}(t+1)\sum_{i=0}^{t+1}\left(\prod_{l=i}^{t} A(l)\right)\right. \\ &\quad \left. \times R_w^t \left(\prod_{l=i}^{t} A(l)\right)^T \mathbf{c}^T(t+1) + R_v^{t+1}\right], \end{aligned}$$

where the last term is independent of control.

Therefore, the minimum of the index for time instance $t+1$ is achieved if and only if the first term on the right-hand side of the above equation is zero. This means that $\{u_k(i), i = 0, 1, \dots, t, k = 1, 2, \dots\}$ is optimal, if $u_k(i) \underset{k\to\infty}{\to} u_d(i)$, $i = 0, 1, \dots, t$. The proof is completed. □

## 3. ILC algorithm and its convergence

In the last section, the optimal control has been characterised, but it cannot be actually used, because the system information is unknown. Thus, the following P-type update law with decreasing learning gain is proposed,

$$\begin{aligned} u_{k+1}(t) = u_k(t) + a_k I_{\{(t+1)\notin \mathcal{M}_k\}} \\ \times (y_d(t+1) - y_k(t+1)), \end{aligned} \tag{4}$$

where $a_k$ is the decreasing gain such that $a_k > 0$, $a_k \to 0$, $\sum_{k=0}^{\infty} a_k = \infty$, $\sum_{k=0}^{\infty} a_k^2 < \infty$, and $a_j = a_k(1 + O(a_k))$, $\forall j = k-K+1, \dots, k-1$, $k$ as $k \to \infty$, where $K$ is defined in Assumption 2. It is clear that $a_k = \frac{1}{k+1}$ meets these requirements. Besides, $I_{\{\text{event}\}}$ is an indicator function meaning that it equals 1 if the event indicated in the bracket is fulfilled, and 0 if the event does not hold.

In order to analyse the convergence of the proposed update law, we first rewrite the system into a super-vector form by the so-called lifting technique as follows,

$$\begin{aligned} U_k &= [u_k(0), u_k(1), \dots, u_k(N-1)]^T \in \mathbb{R}^N \\ Y_k &= [y_k(1), y_k(2), \dots, y_k(N)]^T \in \mathbb{R}^N. \end{aligned}$$

Let $H$ be defined by Equation (5),

$$H = \begin{bmatrix} \mathbf{c}^+\mathbf{b}(0) & 0 & \cdots & 0 \\ \mathbf{c}(2)A(1)\mathbf{b}(0) & \mathbf{c}^+\mathbf{b}(1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{c}(N)\prod_{l=1}^{N-1} A(l)\mathbf{b}(0) & \cdots & \cdots & \mathbf{c}^+\mathbf{b}(N-1) \end{bmatrix} \tag{5}$$

then one has the following relationship between the input and output:

$$Y_k = HU_k + Y_d^0 + W_k, \tag{6}$$

where $Y_d^0 = [(\mathbf{c}(1)A(0)x_d(0))^T, (\mathbf{c}(2)A(1)A(0)x_d(0))^T, \dots, (\mathbf{c}(N)\cdot\prod_{l=0}^{N-1} A(l)\cdot x_d(0))^T]^T$ is the response to initial conditions. The stochastic noise term $W_k$ is expressed by

$$W_k = \begin{bmatrix} \mathbf{c}(1)\sum_{j=0}^{1}(\prod_{l=j}^{0} A(l))w_k(j) + v_k(1) \\ \mathbf{c}(2)\sum_{j=0}^{2}(\prod_{l=j}^{1} A(l))w_k(j) + v_k(2) \\ \vdots \\ \mathbf{c}(N)\sum_{j=0}^{N}(\prod_{l=j}^{N-1} A(l))w_k(j) + v_k(N) \end{bmatrix}. \tag{7}$$

Noticing Assumptions 3 and 4, the noise $W_k$ is considered as a zero-mean Gaussian process noise with covariance matrix $Q$, i.e. $W_k \sim N(0, Q)$. Besides, it is noted that

$$Y_d = HU_d + Y_d^0, \tag{8}$$

where $Y_d$ and $U_d$ are defined similar to $Y_k$ and $U_k$ by replacing $k$ with $d$.

For the lifting model (6), the update law could be lifted as

$$U_{k+1} = U_k + a_k \Gamma_k (Y_d - Y_k), \tag{9}$$

where $\Gamma_k$ is defined in Equation (10)

$$\Gamma_k = \begin{bmatrix} I_{\{1 \notin \mathcal{M}_k\}} & 0 & \cdots & 0 \\ 0 & I_{\{2 \notin \mathcal{M}_k\}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{\{N \notin \mathcal{M}_k\}} \end{bmatrix}. \tag{10}$$

The following theorem shows the convergence property of the proposed update law.

**Theorem 1:** *For the system* (1) *and index* (2)*, assume Assumptions 1–4 hold, then the control sequence* $\{u_k(t)\}$ *given by ILC update law* (4) *is optimal according to Lemma 1. In other words,* $u_k(t)$ *converges to* $u_d(t)$ *a.s. as* $k \to \infty$ *for any* $t \in \{0, 1, 2, \ldots, N-1\}$.

**Remark 1:** In fact, the update law proposed is the stochastic approximation algorithm in essence (Chen, 2002). However, because of the existence of random packet loss, the learning matrix $\Gamma_k$ is no longer of full rank. Thus, the original convergence result of stochastic approximation cannot be applied here directly. More detailed analysis is required.

**Proof:** By introducing Equations (6) and (8) into Equation (9), one has

$$\begin{aligned} U_{k+1} &= U_k + a_k \Gamma_k (Y_d - Y_k) \\ &= U_k + a_k \Gamma_k (HU_d + Y_d^0 - HU_k - Y_d^0 - W_k). \end{aligned}$$

Then subtracting both sides of the above equation from $U_d$, one has

$$\delta U_{k+1} = (I - a_k \Gamma_k H)\delta U_k + a_k \Gamma_k W_k, \tag{11}$$

where $\delta U_k \triangleq U_d - U_k$.

Noticing that $\Gamma_k$ is a diagonal matrix and $H$ is a lower triangular matrix, it is obvious that $\Gamma_k H$ is a lower triangular matrix, thus the eigenvalues of $\Gamma_k H$ are its diagonal elements. By the definition of $\Gamma_k$ and $H$, one has that all eigenvalues of $\Gamma_k H$ are positive or zero. Moreover, any eigenvalue of $\Gamma_k H$ equals zero if and only if the corresponding packet is lost.

Now let us remove all the terms $I_{\{t+1 \notin \mathcal{M}_k\}}$ from $\Gamma_k$ and denote the left matrix as $\Gamma$. Then it is easy to find that the eigenvalues of $\Gamma H$ are positive. Define $P = \int_0^\infty e^{-(\Gamma H)^T t} e^{-(\Gamma H) t} dt$, by simple calculations one can have

$$P(\Gamma H) + (\Gamma H)^T P = I.$$

Noticing the differences between $\Gamma_k$ and $\Gamma$, one could deduce that

$$P(\Gamma_k H) + (\Gamma_k H)^T P \geq 0.$$

By Assumption 2, the summation $\sum_{k=i}^{i+K-1} \Gamma_k H$ is lower triangular matrix whose eigenvalues are positive for any $i \geq 0$, which further implies that there exists some positive constant $\beta > 0$ such that

$$P\left(\sum_{k=i}^{i+K-1} \Gamma_k H\right) + \left(\sum_{k=i}^{i+K-1} \Gamma_k H\right)^T P > \beta I, \ \forall i \geq 0. \tag{12}$$

Define

$$\Phi_{i,j} \triangleq (I - a_i \Gamma_i H) \ldots (I - a_j \Gamma_j H), \, i \geq j, \, \Phi_{i,i+1} \triangleq I. \tag{13}$$

For any $i \geq j + K$, one has

$$\begin{aligned} &\Phi_{i,j}^T P \Phi_{i,j} \\ &= \Phi_{i-1,j}^T \left(I - a_i (\Gamma_i H)^T\right) P (I - a_i(\Gamma_i H)) \Phi_{i-1,j} \\ &= \Phi_{i-K,j}^T (I - a_{i-K+1}(\Gamma_{i-K+1} H)^T) \ldots (I - a_i(\Gamma_i H)^T) P \\ &\quad \times (I - a_i(\Gamma_i H)) \ldots (I - a_{i-K+1}(\Gamma_{i-K+1} H)) \Phi_{i-K,j} \\ &= \Phi_{i-K,j}^T \left[ P - \left( \sum_{k=i-K+1}^{i} a_k (\Gamma_k H)^T P \right.\right. \\ &\quad \left.\left. + \; P \sum_{k=i-K+1}^{i} a_k (\Gamma_k H) \right) + o(a_i) \right] \Phi_{i-K,j} \\ &= \Phi_{i-K,j}^T \left\{ P - a_i \left[ \sum_{k=i-K+1}^{i} (\Gamma_k H)^T P \right.\right. \\ &\quad \left.\left. + \; P \sum_{k=i-K+1}^{i} (\Gamma_k H) \right] + o(a_i) \right\} \Phi_{i-K,j}, \end{aligned}$$

where for the last equality, the condition $a_j = a_k(1 + O(a_k)), \forall j = k - K + 1, \ldots, k-1, k$ is involved.

Noticing that $0 < a_i < 1$ for sufficiently large $i$, then by Equation (12), we have

$$\begin{aligned} &\Phi_{i,j}^T P \Phi_{i,j} \\ &\leq \Phi_{i-K,j}^T (P - a_i \beta I + o(a_i)) \Phi_{i-K,j} \end{aligned}$$

$$\le \Phi_{i-K,j}^T P^{\frac{1}{2}}(I - a_i\beta P^{-1} + o(a_i))P^{\frac{1}{2}}\Phi_{i-K,j}$$
$$\le \Phi_{i-K,j}^T P^{\frac{1}{2}}\left(I - \frac{\beta}{K}P^{-1}\sum_{l=i-K+1}^{i} a_l + o(a_i)\right)P^{\frac{1}{2}}\Phi_{i-K,j}$$
$$\le \left(1 - \frac{\beta}{K}\lambda_{\min}(P^{-1})\sum_{l=i-K+1}^{i} a_l + o(a_i)\right)\Phi_{i-K,j}^T P\Phi_{i-K,j}$$
$$\le \exp\left(-c\sum_{l=i-K+1}^{i} a_l\right)\Phi_{i-K,j}^T P\Phi_{i-K,j}$$

for sufficiently large $j$, where $c$ is a positive constant and $\lambda_{\min}(M)$ denotes the minimum eigenvalue of a matrix $M$.

Then, for sufficiently large $j$, say, for $j > j_0$ and $i > j + K$, we have

$$\Phi_{i,j}^T P\Phi_{i,j} \le c_1 \exp\left(-c\sum_{l=j}^{i} a_l\right) I,$$

where $c_1 > 0$ is a suitable constant, which further, by noticing the definition of $P > 0$, implies

$$\|\Phi_{i,j}\| \le c_2 \exp\left(-\frac{c}{2}\sum_{l=j}^{i} a_l\right),$$

where $c_2 > 0$ is another suitable constant. Then for $\forall i > j_0 + K$, $\forall j > 0$, we have

$$\|\Phi_{i,j}\| = \|\Phi_{i,j_0}\|\|\Phi_{j_0-1,j}\| \le c_0 \exp\left(-\frac{c}{2}\sum_{l=j}^{i} a_l\right) \tag{14}$$

for some $c_0 >$ by noticing $\Phi_{j,j+1} \triangleq I$.

From Equation (11), we have

$$\delta U_{k+1} = \Phi_{k,0}\delta U_0 + \sum_{i=0}^{k}\Phi_{k,i+1}a_i\Gamma_i W_i. \tag{15}$$

Comparing with Lemma 3.3.1 in Chen (2002), it is found that Equations (14) and (15) correspond to (3.1.8) and (3.1.14) of that lemma, respectively. Thus, the rest steps can be carried out along the lines of the proof of Lemma 3.1.1 in Chen (2002). The proof of the theorem is completed. □

**Remark 2:** As one can see, the initial condition, Assumption 4, is formed as a normal distributed random variable. On the other hand, if the initial state is asymptotically available, in other words, $x_k(0) \to x_d(0)$ as $k \to \infty$, Theorem 1 is still valid with a similar proof.

## 4. Extension to nonlinear system

In this section, the linear time-varying model (1) is extended to the following affine nonlinear case with measurement noise

$$\begin{aligned} x_k(t+1) &= f(t, x_k(t)) + \mathbf{b}(t, x_k(t))u_k(t) \\ y_k(t) &= \mathbf{c}(t)x_k(t) + v_k(t), \end{aligned} \tag{16}$$

where $f(t, x_k(t))$ and $\mathbf{b}(t, x_k(t))$ are continuous functions. The control purpose is same as the linear case, see index (2).

The following assumptions are needed.

**Assumption 5:** *The tracking target $y_d(t)$ is realisable in the sense that there exist $u_d(t)$ and $x_d(0)$ such that*

$$\begin{aligned} x_d(t+1) &= f(t, x_d(t)) + \mathbf{b}(t, x_d(t))u_d(t) \\ y_d(t) &= \mathbf{c}(t)x_d(t). \end{aligned} \tag{17}$$

**Assumption 6:** *The functions $f(\cdot,\cdot)$ and $\mathbf{b}(\cdot,\cdot)$ are continuous with respect to the second argument.*

**Assumption 7:** *The initial values can asymptotically be precisely reset in the sense that $x_k(0) \to x_d(0)$ as $k \to \infty$.*

The system noise is removed in the affine nonlinear system (16), thus the noise assumption Assumption 3 is changed as follows.

**Assumption 8:** *For each t, the measurement noise $\{v_k(t)\}$ is a sequence of iid random variables with $\mathbb{E}v_k(t) = 0$, $\sup_k \mathbb{E}v_k^2(t) < \infty$, and $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} v_k^2(t) = R_v^t$, a.s., where $R_v^2$ is unknown.*

For simplicity of writing, let us set $f_k(t) = f(t, x_k(t))$, $f_d(t) = f(t, x_d(t))$, $\mathbf{b}_k(t) = \mathbf{b}(t, x_k(t))$, $\mathbf{b}_d(t) = \mathbf{b}(t, x_d(t))$, $\delta u_k(t) = u_d(t) - u_k(t)$, $\delta f_k(t) = f_d(t) - f_k(t)$, $\delta\mathbf{b}_k(t) = \mathbf{b}_d(t) - \mathbf{b}_k(t)$, $\mathbf{c}^+\mathbf{b}_k(t) = \mathbf{c}(t+1)\mathbf{b}(t, x_k(t))$.

**Lemma 2:** *Assume Assumptions 17–7 hold for system (16). If $\lim_{k\to\infty}\delta u_k(s) = 0$, $s = 0, 1, \ldots, t$, then at time $t+1$, $\|\delta x_k(t+1)\| \underset{k\to\infty}{\to} 0$, $\|\delta f_k(t+1)\| \underset{k\to\infty}{\to} 0$, $\|\delta\mathbf{b}_k(t+1)\| \underset{k\to\infty}{\to} 0$.*

**Proof:** The proof of this lemma can be carried out by induction along the time axis $t$. By Equations (16) and (17),

$$\begin{aligned} \delta x_k(t+1) &= f_d(t) - f_k(t) + \mathbf{b}_d(t)u_d(t) - \mathbf{b}_k(t)u_k(t) \\ &= \delta f_k(t) + \delta\mathbf{b}_k(t)u_d(t) + \mathbf{b}_k(t)\delta u_k(t). \end{aligned} \tag{18}$$

Thus, for $t = 0$, noticing Assumptions 6 and 7, one has

$$\begin{aligned} \delta f_k(0) &= f_d(0) - f_k(0) \underset{k\to\infty}{\to} 0 \\ \delta\mathbf{b}_k(0) &= \mathbf{b}_d(0) - \mathbf{b}_k(0) \underset{k\to\infty}{\to} 0, \end{aligned}$$

which imply that the first two terms at the right-hand side of Equation (18) tends to zero as $k \to \infty$. Since

$$\|\mathbf{b}_k(0)\| \le \|\mathbf{b}_d(0)\| + \|\delta \mathbf{b}_k(0)\|,$$

it follows that $\mathbf{b}_k(0)$ is bounded. Thus, if $\delta u_k(0) \underset{k\to\infty}{\to} 0$, then the third term at the right-hand side of Equation (18) also tends to zero. It further implies that $\delta x_k(1) \underset{k\to\infty}{\to} 0$ and then by Assumption 6 again, $\delta f_k(1) \underset{k\to\infty}{\to} 0$ and $\delta \mathbf{b}_k(1) \underset{k\to\infty}{\to} 0$. That is, the conclusion are valid for $t = 0$.

Now assume the conclusions of the lemma are true for $s = 0, 1, \dots, t-1$, it suffices to show that the conclusions hold for $t$, i.e., $\|\delta x_k(t+1)\| \underset{k\to\infty}{\to} 0$, $\|\delta f_k(t+1)\| \underset{k\to\infty}{\to} 0$, $\|\delta \mathbf{b}_k(t+1)\| \underset{k\to\infty}{\to} 0$. This could be done using the same argument as above. This completes the proof. □

Based on Lemma 2, the conclusion of Lemma 1 also holds for nonlinear system (16) with Assumptions 1–4 replaced by Assumption 1 and Assumptions 17–8, by using similar steps of Lemma 1. Thus. the conclusions and proofs are no longer copied here.

The update law (4) is adopted for the nonlinear case. For any fixed time $t$ from Equation (4), one has

$$\begin{aligned}\delta u_{k+1}(t) = \delta u_k(t) - a_k I_{\{(t+1)\notin \mathcal{M}_k\}}[&\mathbf{c}^+\mathbf{b}_k(t)\delta u_k(t)\\ &+ \varphi_k(t) - \upsilon_k(t+1)],\end{aligned} \tag{19}$$

where

$$\varphi_k(t) = \mathbf{c}^+\delta f_k(t) + \mathbf{c}^+\delta \mathbf{b}_k(t) u_d(t). \tag{20}$$

We have the following convergence results.

**Theorem 2:** *For system* (16) *and index* (2)*, assume Assumptions* 1, 2 *, and Assumptions* 17–8 *hold, then the control sequence* $\{u_k(t)\}$ *given by the ILC update law* (4) *is optimal. In other words,* $u_k(t)$ *converges to* $u_d(t)$ *a.s. as* $k \to \infty$ *for any* $t \in \{0, 1, \dots, N-1\}$.

**Proof:** As a result of the existence of nonlinear functions $f_k(t)$ and $\mathbf{b}_k(t)$, it is hard to formulate the super-vector form of affine nonlinear system (16) just as the linear case. Thus, the proof of Theorem 1 cannot be directly applied here. Instead the proof is carried out by induction along the time axis $t$ with similar techniques of the proof of Theorem 1.

For $t = 0$, the algorithm (4) is written as

$$\begin{aligned}\delta u_{k+1}(0) = (1 - a_k I_{\{1\notin \mathcal{M}_k\}}&\mathbf{c}^+\mathbf{b}_k(0))\delta u_k(0)\\ &- a_k I_{1\notin \mathcal{M}_k}\varphi_k(0) + a_k I_{1\notin\mathcal{M}_k}\upsilon_k(1).\end{aligned} \tag{21}$$

Since $\mathbf{b}_k(0)$ is continuous in the initial state by Assumption 6, one has $\mathbf{b}_k(0) \underset{k\to\infty}{\to} \mathbf{b}_d(0)$ by Assumption 7 and $\mathbf{c}^+\mathbf{b}_k(0)$ converges to a positive constant by Assumption 1. Therefore, by Assumption 2 it follows

$$\sum_{k=i}^{i+K-1}\left(-I_{\{1\notin\mathcal{M}_k\}}\mathbf{c}^+\mathbf{b}_k(0)\right) < -\gamma, \quad \gamma > 0 \tag{22}$$

for all sufficient large $i$.

Set $\phi_{i,j} \triangleq (1 - a_i I_{\{1\notin\mathcal{M}_i\}}\mathbf{c}^+\mathbf{b}_i(0))\dots(1 - a_j I_{\{1\notin\mathcal{M}_j\}}\mathbf{c}^+\mathbf{b}_j(0))$, $i \ge j$, $\phi_{i,i+1} \triangleq 1$. It is clear that $1 - a_j I_{\{1\notin\mathcal{M}_j\}}\mathbf{c}^+\mathbf{b}_j(0) > 0$ for all large enough $j$, say, $j \ge j_0$. Then for any $i \ge j + K$, $j \ge j_0$ by Equations (21) and (22), one has that

$$\begin{aligned}\phi_{i,j} &= \phi_{i-K,j}\left(1 - a_i \sum_{k=i-K+1}^{i} I_{\{1\notin\mathcal{M}_k\}}\mathbf{c}^+\mathbf{b}_k(0) + o(a_i)\right)\\ &\le \phi_{i-K,j}(1 - \gamma a_i + o(a_i))\\ &= \phi_{i-K,j}\left(1 - \frac{\gamma}{K}\sum_{k=i-K+1}^{i} a_k + o(a_i)\right)\\ &\le \exp\left(-c\sum_{k=i-K+1}^{i} a_k\right)\phi_{i-K,j} \quad \text{with } c > 0.\end{aligned}$$

It follows from here that $\phi_{i,j} \le c_3 \exp(-\frac{c}{2}\sum_{k=j}^{i} a_k)$, $\forall j \ge j_0$ for some $c_3 > 0$, and hence there is a $c_4$ such that

$$|\phi_{i,j}| \le c_4 \exp\left(-\frac{c}{2}\sum_{k=j}^{i} a_k\right), \quad \forall i \ge j + K, j \ge j_0.$$

Therefore, for $\forall i \ge j_0 + K$, $\forall j \ge 0$ one has

$$|\phi_{i,j}| \le |\phi_{i,j_0}||\phi_{j_0-1,j}| \le c_5 \exp\left(-\frac{c}{2}\sum_{k=j}^{i} a_k\right) \tag{23}$$

for some $c_5 > 0$.

From Equation (21) it follows that

$$\begin{aligned}\delta u_{k+1}(0) = \phi_{k,0}\delta u_0(0) &- \sum_{j=0}^{k}\phi_{k,j+1}a_k I_{1\notin\mathcal{M}_k}\varphi_k(0)\\ &+ \sum_{j=0}^{k}\phi_{k,j+1}a_k I_{1\notin\mathcal{M}_k}\upsilon_k(1),\end{aligned} \tag{24}$$

where the first term at the right-hand side of the above equation tends to zero as $k \to \infty$ because of Equation (23). By Assumptions 6 and 7, it is clear that $\varphi_k(0) \underset{k\to\infty}{\to} 0$. By Assumption 8, it follows that

$$\sum_{k=1}^{\infty} a_k I_{1\notin\mathcal{M}_k}\upsilon_k(1) < 0.$$

Thus, the last two terms at the right-hand side of Equation (24) also tend to zero as $k \to \infty$ by using similar proof steps to Lemma 3.1.1 of Chen (2002). Thus, the optimality of the control sequence $\delta u_k(0)$ has been proved.

Inductively, assume that the optimality takes place for $t = 0, 1, \dots, s-1$. It is now to show the validity for $t = s$.

By the inductive assumption, one has $\delta u_k(t) \underset{k\to\infty}{\to} 0$, $t = 0, 1, \dots, s-1$, and by Lemma 2,

$$\delta x_k(s) \underset{k\to\infty}{\to} 0,\ \delta f_k(s) \underset{k\to\infty}{\to} 0,\ \delta b_k(s) \underset{k\to\infty}{\to} 0.$$

Then it follows that $\phi_k(s) \underset{k\to\infty}{\to} 0$. As for the case $t = 0$, a similar treatment leads to $u_k(s) \underset{k\to\infty}{\to} u_d(s)$. This proves optimality of control at $t = s$ and completes the proof of the theorem. □

## 5. Illustrative simulations

We consider two examples, one of which is a second-order linear system and the other one is a second-order affine nonlinear system as follows,

(1) Linear Case:

$$x_k(t+1) = \begin{bmatrix} 1 & 0.1 \\ 0.05 & 1.1 \end{bmatrix} x_k(t) + \begin{bmatrix} 0.45 \\ 0.6 \end{bmatrix} u_k(t) + w_k(t+1)$$
$$y_k(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x_k(t) + v_k(t).$$

(2) Nonlinear Case:

$$x_k^{(1)}(t+1) = 1.12 x_k^{(1)}(t) + 0.3\sin(x_k^{(2)}(t)) + 0.45 u_k(t)$$
$$x_k^{(2)}(t+1) = 0.2\cos(x_k^{(1)}(t)) + 1.1 x_k^{(2)}(t) + 0.5 u_k(t)$$
$$y_k(t) = x_k^{(1)}(t) + x_k^{(2)}(t) + v_k(t),$$

where $x_k^{(1)}(t)$ and $x_k^{(2)}(t)$ denote the first and second dimension of $x_k(t)$, respectively.

For simple illustration, let $N = 10$, and the noises are assumed zero-Gaussian distributed, i.e., $w_k(t) \sim N(0, 0.05^2 I_2)$, $v_k(t) \sim N(0, 0.1^2)$.

In order to simulate measurement loss, let us fix $K = 4$. Iteration steps are separated into groups of four successive iterations, i.e., $\{1, 2, 3, 4\}$, $\{5, 6, 7, 8\}, \dots$, and randomly select one iteration from each group for each $t$, say, 1, 6, 9, … for example. For these selected steps, the control is not updated. Figure 2 is an arbitrary illustration of packet loss step for the first 100 iterations, where 0 denote that the packet is lost while 1 means that the packet is successfully transmitted.

The reference trajectory is $y_d(t) = 2t$. The initial control action is simply given as $u_0(t) = 0$, $\forall t$. The learning gain chooses $a_k = \frac{1}{k+1}$. The algorithm has run 300 iterations. The output of the 300th iteration is shown in Figures 3 and 4 for linear and nonlinear cases, respectively, where the solid lines are the reference signals and the dashed line with

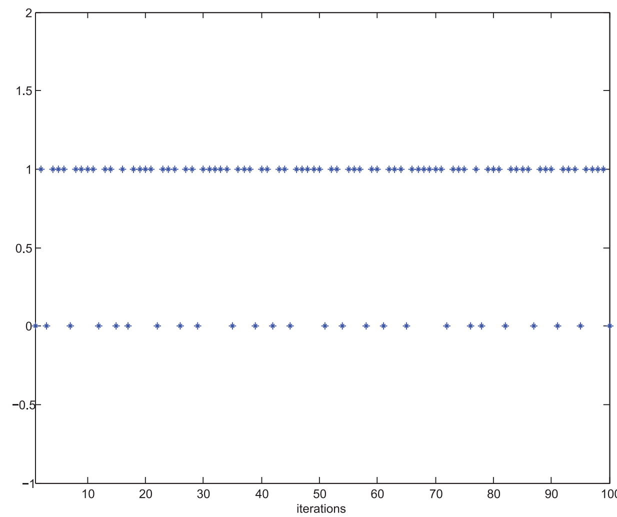


Figure 2. Illustration of packet loss.

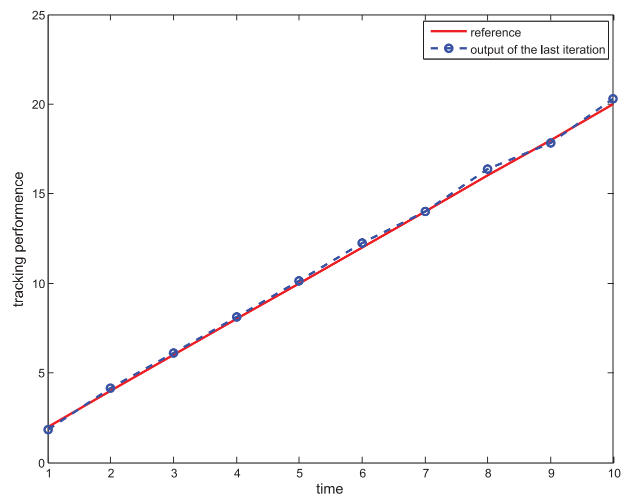


Figure 3. $y_{300}(t)$ vs. $y_d(t)$ for the linear case.

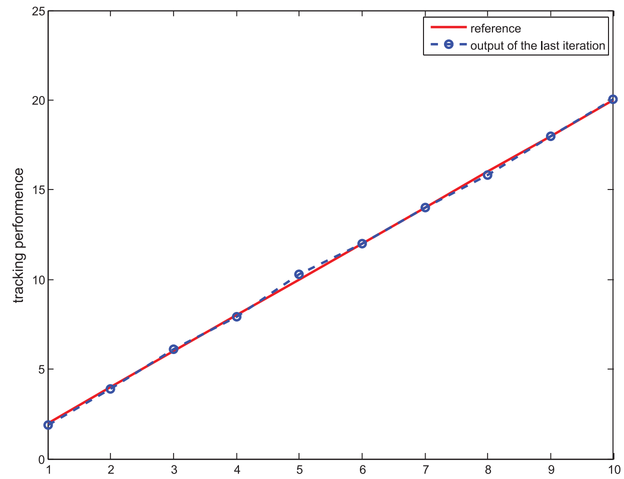


Figure 4. $y_{300}(t)$ vs. $y_d(t)$ for the nonlinear case.

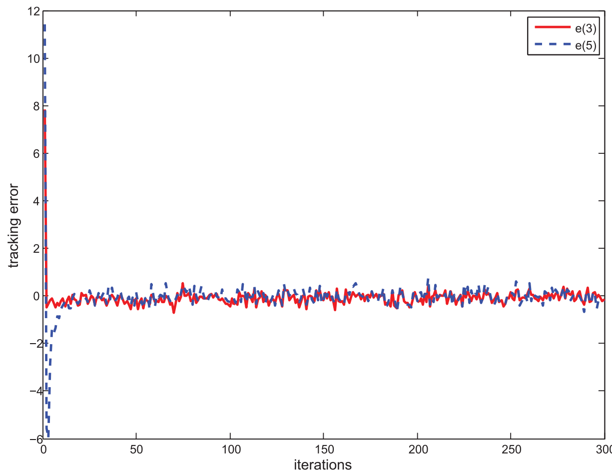


Figure 5. Tracking errors $e_k(t)$ at $t = 3$ and $t = 5$ for the linear case.

cycles denote the output $y_{300}(t)$. As one could see, the actual output could track the desired reference trajectory with quite small tracking errors, which are mainly involved by the stochastic noises. These verify the effectiveness and precision of the proposed ILC algorithm under random packet losses and stochastic noises.

The tracking errors at $t = 3$ and $t = 5$ as examples are demonstrated in Figures 5 and 6 for linear and nonlinear cases, respectively. The tracking errors along the iterations at the other time instances are similar to those in Figures 5 and 6. As one can see, the tracking errors decay to zero in few iterations and then fluctuate at zero in a very narrow range. The fluctuations are caused by the stochastic noises at each iteration. That is to say, if the stochastic noises are eliminated from the actual tracking error, the left would decrease to zero.

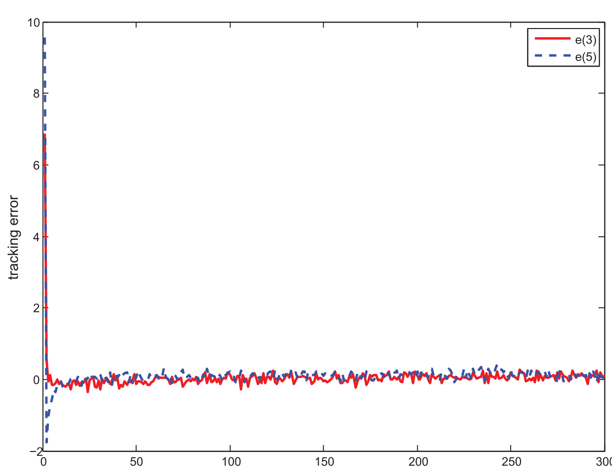


Figure 6. Tracking errors $e_k(t)$ at $t = 3$ and $t = 5$ for the nonlinear case.

We also make the simulations for the case that the random packet loss is modelled as a binary Bernoulli random variable $\gamma$, which is distributed as $P(\gamma = 1) = 0.75$ and $P(\gamma = 0) = 0.25$. The simulation results are similar to Figures 3– 6, which implies that our algorithm has a good and robust performance for different random packet loss forms. For the sake of saving space, these figures are not presented. The detailed convergence analysis will be given in another paper.

## 6. Conclusions

The ILC is considered for stochastic systems with random packet losses in transmission. The random packet loss in this paper is modelled by an arbitrary stochastic sequence with bounded length requirement. The P-type control update algorithm is proposed for the SISO linear time-varying stochastic system and the convergence with probability one of control to the optimal one is established. Then similar results are also extended to the affine nonlinear system with measurement noises. Illustrative simulations verify the theoretical analysis. For further study, the ILC for more general nonlinear system with random packet loss is an interesting topic. Besides, it is also attractive to consider more models of random packet loss such as Markov chain. Furthermore, it is only the measurement loss that is considered in this paper, thus it is worth taking control loss or both of them into account.

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