Reliable $H_{\infty}$ control for uncertain nonlinear discrete-time systems subject to multiple intermittent faults in sensors and/or actuators

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Abstract

This study focuses on passive fault-tolerant control for a class of uncertain nonlinear discrete-time systems subject to multiple intermittent faults. The considered intermittent faults are assumed to be additive ones in sensors and/or actuators. To achieve fault-tolerant control, a dynamic output-feedback controller is designed such that the closed-loop system remains stable and satisfies acceptable performance, even when there are parameter uncertainties, nonlinearities of specific type, and multiple additive intermittent sensor and/or actuator faults. The linear matrix inequality method is employed to obtain sufficient conditions for achieving fault tolerance and ensuring the prescribed $H_{\infty}$ performance index. Finally, the effectiveness of the proposed method is demonstrated by simulation examples.

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1. Introduction

Fault-tolerant control (FTC) aims to ensure that a controlled system retains an acceptable performance even if some system components (including sensors and actuators) malfunction. With the growing scale and complexity of industrial processes, the advent of faults may endanger system security and cause serious economic loss, especially in safety-critical systems, such as chemical plants, aircrafts, spacecraft, and nuclear power plants. The engineering significance of

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FTC has been well recognized, and a number of practical systems have adopted the fault-tolerant strategy, thus creates ample scope for FTC studies within the system control community [1–4]. Usually, all reported FTC studies can be divided into two categories: active FTC (AFTC) [5–11] and passive FTC (PFTC) [12–18]. To achieve the purpose of AFTC, fault detection and diagnosis, which itself also draws considerable concern in control community [19,20], is necessary to provide real-time fault information to reconfigure the controller. In contrast, the design of PFTC relies on a priori knowledge of failure information (e.g. location, amplitude).

System faults are the focus of FTC design, and they are present in different forms within systems. In most cases, the term “fault” in the above-mentioned papers refers to a permanent fault (PF), which tends to remain faulty and may even worsen without intervention. However, many faults are not permanent, and a few types of such non-permanent faults are called intermittent faults (IFs). IFs exist widely in practical systems, including electronic equipment [21], spacecraft [22], mechanical devices [23], and communication systems [24], and they threaten system reliability and security. The intermittent nature of a fault is reflected in the fact that it can recover without any corrective action. Therefore, as compared with the PF, the occurrence of the IF exhibits special properties such as randomness, intermittence, and repeatability [25,26].

Owing to the existence of these special properties, traditional methods for addressing PFs cannot be applied to address IFs directly, regardless of the fault diagnosis technique or fault-tolerant scheme employed. From the viewpoint of FTC, the first hindrance is the lack of effective mathematical description. To solve the problem in the FTC framework, usually, a PF is described as an analytical model in deterministic form. However, it is difficult to depict an IF using a deterministic analytical form, because a suitable IF model must be able to switch frequently and randomly between fault occurrence and disappearance. In fact, the majority of IFs occur randomly, thus necessitating the use of stochastic variables to describe them [22,27–29]. To the best of the authors’ knowledge, few studies on the FTC of multiple IFs have been reported.

Motivated by the aforementioned state of the art, this study investigates PFTC problem for a class of uncertain nonlinear discrete-time systems subject to multiple IFs, where nonlinearities are assumed to satisfy the sector-bounded condition and parameter uncertainty is norm bounded. The multiple IFs, represented by a set of independent Bernoulli-distributed stochastic variables, are formulated as probabilistic events occurring on each sensor and/or actuator of the system. The aim is to design a dynamic output-feedback controller such that the closed-loop system maintains asymptotical stability and satisfies a given $H_\infty$ performance index. Using the linear matrix inequality (LMI) approach, a sufficient condition for controller existence is obtained. Finally, two examples are provided to illustrate the effectiveness of the proposed scheme.

The remainder of this paper is organized as follows. In Section 2, the mathematical model of the considered system is formulated, and some preliminaries are given. Section 3 presents the main results of PFTC of an uncertain nonlinear discrete-time system against multiple IFs, and scenarios of sensor faults, actuator faults, and both of them are investigated in sequence. In Section 4, two simulation examples are provided to demonstrate the effectiveness of the proposed method. Finally, the conclusions of this study are summarized in Section 5.

Notations: Throughout the paper, for real symmetric matrices, the notation $X \succ Y$ indicates that the matrix $X - Y$ is positive definite. $\mathbb{R}^n$ represents $n$-dimensional Euclidean space, while $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices. The superscript $T$ denotes transpose. $\mathbb{E}\{\cdot\}$ represents the expectation of event $\cdot$. $\text{Prob}\{\cdot\}$ represents the occurrence probability of event $\cdot$. $\text{diag}\{M_1, M_2, \ldots, M_n\}$ denotes a block diagonal matrix composed of square matrices $M_1$, $M_2$, $\ldots$, $M_n$. $l_2[0,\infty)$ is the space of square summable vectors. The symbol $\ast$ is used to
represent the corresponding transposed block in the symmetry block matrix. \( I \) denotes an identity matrix with appropriate dimensions.

2. Problem formulation and preliminaries

Consider a class of uncertain nonlinear discrete-time systems subject to multiple intermittent faults:

\[
\begin{aligned}
    x(k+1) &= (A + \Delta A)x(k) + B(u(k) + \Xi_a(k)f_a(k)) + h(x(k)) \\
y(k) &= C_1x(k) + \Xi_s(k)f_s(k) \\
z(k) &= C_2x(k)
\end{aligned}
\]

where \( x(k) \in \mathbb{R}^n \) is the system state, \( u(k) \in \mathbb{R}^m \) is the control input, \( y(k) \in \mathbb{R}^p \) is the measured output, and \( z(k) \in \mathbb{R}^r \) is the desired controlled output. \( A, B, C_1, \) and \( C_2 \) are known system matrices of appropriate dimensions. The unknown matrix \( \Delta A \) represents modeling uncertainty, and is assumed to satisfy \( \Delta A = MFN \), where \( M \) and \( N \) are known real matrices with appropriate dimensions, and \( F \) follows the constraint \( FF^T \preceq I \).

In addition, \( h(x(k)) \) is a nonlinear function satisfying the following sector-bounded condition:

\[
[h(x(k)) - T_1x(k)]^T[h(x(k)) - T_2x(k)] \leq 0 \quad \forall x(k) \in \mathbb{R}^n,
\]

where \( T_1, T_2 \in \mathbb{R}^{n \times n} \) are known real matrices.

The considered faults are reflected in the terms \( \Xi_s(k)f_s(k) \) and \( \Xi_a(k)f_a(k) \), in which the vectors \( f_s(k) \in \mathbb{R}^p \) and \( f_a(k) \in \mathbb{R}^m \) represent multiple sensor and actuator faults, respectively. The stochastic variables \( \pi_i(k)(i = 1, \ldots, p) \) and \( \tau_j(k)(j = 1, \ldots, m) \) in diagonal matrices \( \Xi_s(k) = \text{diag}\{\pi_1(k), \ldots, \pi_p(k)\} \) and \( \Xi_a(k) = \text{diag}\{\tau_1(k), \ldots, \tau_m(k)\} \) account for the occurrence of IFs in the \( i \)th sensor signal and the \( j \)th actuator signal, obeying Bernoulli distribution and taking values of 1 or 0 as follows:

\[
\begin{aligned}
    \text{Prob}\{\pi_i(k) = 1\} &= \pi_i \\
    \text{Prob}\{\pi_i(k) = 0\} &= 1 - \pi_i
\end{aligned}
\]

\[
\begin{aligned}
    \text{Prob}\{\tau_j(k) = 1\} &= \tau_j \\
    \text{Prob}\{\tau_j(k) = 0\} &= 1 - \tau_j
\end{aligned}
\]

here \( \pi_i \in [0, 1] \) \( (i = 1, \ldots, p) \) and \( \tau_j \in [0, 1] \) \( (j = 1, \ldots, m) \) are known constants. In the following, the expectations of \( \Xi_s(k) \) and \( \Xi_a(k) \) are denoted as \( \Xi_s = \text{diag}\{\pi_1, \ldots, \pi_p\} \) and \( \Xi_a = \text{diag}\{\tau_1, \ldots, \tau_m\} \), respectively.

**Remark 1.** IFs exist in many practical systems such as electronic equipment, spacecraft, mechanical devices, and communication systems. Many of these faults are in the nature of randomness. The majority of the random IFs may result from the system itself, such as net congestion in networked systems and electromagnetic interference in electronic systems, and a few of them are caused by the environment. Because it is difficult to describe these faults in a deterministic fashion, stochastic variables are employed widely to describe them [22,27,28,30].

**Remark 2.** In theory, with the proposed IF description, the fault itself can be of any form, such as a square wave, triangular wave, or any other regular or irregular signal. However, in the framework of reliable \( H_\infty \) control, it is assumed that the considered faults are square summable, i.e., \( f_s(k) \) and \( f_a(k) \) belong to \( l_2[0, \infty) \).
This study aims to design a class of dynamic full-order output feedback controllers of the following form:

\[
\begin{aligned}
\dot{x}_c(k+1) &= A_c x_c(k) + B_c y(k) \\
 u(k) &= C_c x_c(k)
\end{aligned}
\]

(3)

where \(x_c(k) \in \mathbb{R}^n\) is the state vector of the controller, and \(A_c, B_c, \) and \(C_c\) are controller matrices with appropriate dimensions to be designed later.

Define

\[
\eta(k) = \begin{bmatrix} x(k) \\ x_c(k) \end{bmatrix}
\]

(4)

Three closed-loop system situations, which correspond to the cases of only sensor faults, only actuator faults, and both these two kinds of faults, respectively, can be formulated by substituting Eqs. (3) and (4) into the corresponding scenario descriptions revised from Eq. (1). A general form of the closed-loop system is summarized as follows:

\[
\begin{aligned}
\eta(k+1) &= \bar{A}_g \eta(k) + \bar{B}_g \Xi_g(k) f_g(k) + E h(D \eta(k)) \\
 z(k) &= \bar{C}_g \eta(k)
\end{aligned}
\]

(5)

where subscript “\(g\)” denotes the general form, and \(\bar{A}_g, \bar{B}_g, \) and \(\bar{C}_g\) are matrices of the closed-loop system.

The study objective is to design dynamic output feedback controller (3) for system (1) with multiple sensor and/or actuator faults, such that the closed-loop system (5) (Fig. 1) meets the following requirements:

(1) Under the condition of \(\Xi_g(k) \equiv 0\), the closed-loop system (5) is asymptotically stable.

(2) Under the zero-initial condition, the controlled output satisfies

\[
\sum_{k=0}^{\infty} \mathbb{E} \{ \| z(k) \|^2 \} < \gamma^2 \sum_{k=0}^{\infty} \mathbb{E} \{ \| \Xi_g(k) f_g(k) \|^2 \}
\]

where \(\gamma > 0\) is a prescribed scalar.

Fig. 1. The structure of the closed-loop system.
Before presenting the main results, it is necessary to introduce some lemmas, which will be used frequently.

**Lemma 1.** [31]: Given constant matrices $S_1$, $S_2$, and $S_3$, where $S_1 = S_1^T$ and $S_2 = S_2^T > 0$, $S_1 + S_3 S_2^{-1} S_3 < 0$ if and only if

$$
\begin{bmatrix}
S_1 & S_3^T \\
S_3 & -S_2
\end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix}
-S_2 & S_3 \\
S_3^T & S_1
\end{bmatrix} < 0.
$$

**Lemma 2.** [32]: Let $M$, $N$, and $F$ be real matrices of appropriate dimensions with $F$ satisfying $FF^T \leq I$. Then, for any scalar $\varepsilon > 0$, one obtains

$$
MFN + (MFN)^T \leq \varepsilon^{-1} MM^T + \varepsilon N^TN.
$$

**Lemma 3.** The sector-bounded condition (2) is equivalent to

$$
\begin{bmatrix}
x(k) \\
h(x(k))
\end{bmatrix}^T \begin{bmatrix}
R_1 & R_2 \\
R_2^T & I
\end{bmatrix} \begin{bmatrix}
x(k) \\
h(x(k))
\end{bmatrix} \leq 0.
$$

Here $R_1 = (T_1^T T_2 + T_2^T T_1)/2$ and $R_2 = -(T_1^T + T_2^T)/2$.

The proof of Lemma 3 is omitted because it is straightforward from Eq. (2).

**Lemma 4.** [33]: Let $W_0(x)$, $W_1(x)$, ..., $W_l(x)$ be quadratic functions of $x \in \mathbb{R}^n$, i.e.,

$$
W_i(x) = x^T \Phi_i x, i = 0, 1, \ldots, l
$$

with $\Phi_i^T = \Phi_i$. Then, the implication

$$
W_1(x) \leq 0, \ldots, W_l(x) \leq 0 \Rightarrow W_0(x) < 0,
$$

holds, if there exist scalars $\tau_1 \geq 0, \tau_2 \geq 0, \ldots, \tau_l \geq 0$ such that

$$
\Phi_0 - \sum_{i=1}^{l} \tau_i \Phi_i < 0.
$$

In the following, reliable $H_{\infty}$ control of an uncertain nonlinear discrete-time system subject to multiple IFs is investigated. Three fault scenarios, sensor faults, actuator faults, and both of them, are considered in sequence.

3. Main results

3.1. Case A: multiple sensor faults

Firstly, consider a class of systems that only involves multiple sensor IFs as follows:

$$
\begin{align*}
x(k + 1) &= (A + \Delta A)x(k) + Bu(k) + h(x(k)) \\
y(k) &= C_1 x(k) + \Xi_s(k) f_s(k) \\
z(k) &= C_2 x(k)
\end{align*}
$$

(6)

where all parameters in Eq. (6) have the same definitions as those in Eq. (1).
Substituting Eqs. (3) and (4) into Eq. (6) yields the following closed-loop system:

\[
\begin{align*}
\eta(k + 1) &= \bar{A}\eta(k) + \bar{B}\left(f_s(k) + Eh(D\eta(k))\right) \\
\quad = \bar{A}\eta(k) + \bar{B}\left(f_s(k) + Eh(D\eta(k))\right), \\
z(k) &= \bar{C}\eta(k)
\end{align*}
\] (7)

where

\[
\bar{A} = \begin{bmatrix} A + \Delta A & BC_c \\ B_c C_1 & A_c \end{bmatrix} = \begin{bmatrix} A & BC_c \\ B_c C_1 & A_c \end{bmatrix} + \begin{bmatrix} M^T & \textbf{0} \end{bmatrix} F \begin{bmatrix} N & \textbf{0} \end{bmatrix} = \bar{A} + \bar{M} F \bar{N},
\]

\[
\bar{B} = \begin{bmatrix} \textbf{0} \\ B_c \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} 0 \\ B_c \Xi_s(k) \end{bmatrix}, \quad E = \begin{bmatrix} I & \textbf{0} \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C_2 & \textbf{0} \end{bmatrix}, \quad D = \begin{bmatrix} I & \textbf{0} \end{bmatrix}.
\]

According to Section 2, the control objective is to design a dynamic output feedback controller of the form (3) such that

1. Under the condition \(\Xi_s(k) = \textbf{0}\), the closed-loop system (7) is asymptotically stable; and
2. Under the zero-initial condition, the controlled output satisfies the following performance index:

\[
\sum_{k=0}^{\infty} \mathbb{E}\left\{\|z(k)\|^2\right\} < \gamma_s^2 \sum_{k=0}^{\infty} \mathbb{E}\left\{\|\Xi_s(k)f_s(k)\|^2\right\}
\] (8)

where \(\gamma_s > 0\) is a prescribed scalar.

**Theorem 1.** For a prescribed scalar \(\gamma_s > 0\), if there exists matrix \(P = P^T > 0\) such that

\[
\begin{bmatrix} -P & \textbf{0} & P\bar{A} & PE & P\bar{B}_2 \\
* & -I & \bar{C} & \textbf{0} & \textbf{0} \\
* & * & -P - \tau D^T R_1 D - \tau D^T R_2 & \textbf{0} & \textbf{0} \\
* & * & * & -\tau I & \textbf{0} \\
* & * & * & * & -\gamma_s^2 \Xi_s \end{bmatrix} < 0
\] (9)

holds, where \(\bar{B}_2 = \begin{bmatrix} \textbf{0} & (B_c \Xi_s)^T \end{bmatrix}^T\), then the closed-loop system (7) is asymptotically stable, and \(H_\infty\) performance index (8) is satisfied.

**Proof.** Define the Lyapunov function as

\[
V(k) = \eta^T(k)P\eta(k). \quad (10)
\]

Under the condition \(\Xi_s(k) = \textbf{0}\), the difference of the Lyapunov function is

\[
\Delta V(k) = V(\eta(k + 1)) - V(\eta(k))
\]

\[
= (\bar{A}\eta(k) + Eh(D\eta(k)))^T P(\bar{A}\eta(k) + Eh(D\eta(k))) - \eta^T(k)P\eta(k)
\]

\[
= \begin{bmatrix} \eta(k) \\
h(D\eta(k)) \end{bmatrix}^T \begin{bmatrix} \bar{A}^T P\bar{A} - P & \bar{A}PE \\
\textbf{0} & E^T PE \end{bmatrix} \begin{bmatrix} \eta(k) \\
h(D\eta(k)) \end{bmatrix}.
\]
Applying Lemma 1 to Eq. (9) implies that
\[
\begin{bmatrix}
A^TPA - P + C^T C & \hat{A}^TP E \\
* & E^T PE
\end{bmatrix} - \begin{bmatrix}
\tau D^T R_1 D & \tau D^T R_2 \\
* & \tau I
\end{bmatrix} < 0.
\]

According to Lemmas 3 and 4, it is straightforward that \(\Delta V(k) < 0\). Hence, the closed-loop system (7) is asymptotically stable.

To evaluate the \(H_\infty\) performance of the closed-loop system, the following index \(J\) is introduced under the zero-initial condition.

\[
J = \sum_{k=0}^{\infty} \mathbb{E}\left\{\|z(k)\|^2 - \gamma_s^2 \|\Xi_s(k)f_s(k)\|^2 + \Delta V(k)\right\} + \mathbb{E}\{V(0)\} - \mathbb{E}\{V(\infty)\}
\]

\[
\leq \sum_{k=0}^{\infty} \mathbb{E}\left\{\|z(k)\|^2 - \gamma_s^2 \|\Xi_s(k)f_s(k)\|^2 + \Delta V(k)\right\}.
\]

We, therefore, construct Eq. (12).

\[
\mathbb{E}\left\{\|z(k)\|^2 - \gamma_s^2 \|\Xi_s(k)f_s(k)\|^2 + \Delta V(k)\right\}
\]

\[
= \mathbb{E}\left\{\frac{\hat{A}\hat{\eta}(k) + Eh(D\hat{\eta}(k)) + \overline{B}_1f_s(k)^T P(\hat{A}\hat{\eta}(k) + Eh(D\hat{\eta}(k)) + \overline{B}_1f_s(k))}{-\hat{\eta}^T(k)P\hat{\eta}(k) + \hat{\eta}^T(k)\overline{C}^T\overline{C}\hat{\eta}(k) - \gamma_s^2 f_s(k)^T \Xi_s f_s(k)}\right\}
\]

\[
= \mathbb{E}\left\{\varphi^T(k)(\varphi(k)\Omega\varphi(k) + \text{diag}\{-P + \overline{C}^T\overline{C}, 0, -\gamma_s^2 \Xi_s\})\varphi(k)\right\}
\]

\[
= \mathbb{E}\left\{\varphi^T(k)\Omega\varphi(k)\right\},
\]

where

\[
\varphi(k) = \begin{bmatrix}
\eta^T(k) & \eta^T(D\eta(k)) & \eta^T f_s(k)
\end{bmatrix}^T,
\]

\[
\phi(k) = \begin{bmatrix}
\overline{A} & \overline{E} & \overline{B}_2
\end{bmatrix}.
\]

Applying Lemma 1 to Eq. (9) implies that

\[
\Omega - \begin{bmatrix}
\tau D^T R_1 D & \tau D^T R_2 & \mathbf{0} \\
* & \tau I & \mathbf{0} \\
* & * & \mathbf{0}
\end{bmatrix} < 0
\]

According to Eq. (13) and using Lemmas 3 and 4 again, it can be inferred that \(\varphi^T(k)\Omega\varphi(k) < 0\), which demonstrates that \(J < 0\). Therefore, inequality (8) holds, which completes the proof.

Obviously, expression (9) is a bilinear matrix inequality (BMI) rather than a LMI. It cannot be solved directly. Hence, the method adopted in [34] is used to solve BMI (9) and derive the controller parameters.
Theorem 2. For given scalar $\gamma_s > 0$, symmetric matrix $X > 0$, and scalar $\varepsilon > 0$, if there exists a symmetric matrix $Y > 0$, real matrices $\hat{A} \in \mathbb{R}^{n \times n}$, $\hat{B} \in \mathbb{R}^{n \times p}$, and $\hat{C} \in \mathbb{R}^{m \times n}$, and a positive scalar $\tau$ such that

$$
\begin{bmatrix}
-X & -I & 0 & AX + \hat{B}\hat{C} & A & I & 0 & M & 0 \\
* & -Y & 0 & \hat{A} & YA + \hat{B}\hat{B}_s^{-1}C_1 & Y & \hat{B} & YM & 0 \\
* & * & -I & C_2X & 0 & 0 & 0 & 0 \\
* & * & * & -X - \tau XR_1X & -I - \tau XR_1 & -\tau XR_2 & 0 & 0 & \varepsilon XN^T \\
* & * & * & * & -Y - \tau R_1 & -\tau R_2 & 0 & 0 & \varepsilon N^T \\
* & * & * & * & * & -\varepsilon l & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -\varepsilon I & 0 \\
* & * & * & * & * & * & * & * & -\varepsilon l 
\end{bmatrix} < 0
$$

holds, then closed-loop system (7) is asymptotically stable and $H_\infty$ performance (8) can be satisfied. Furthermore, the controller matrices are designed as follows:

$$
A_c = S^{-1} \left( \hat{A} - \hat{B}\hat{B}_s^{-1}C_1X - YB\hat{C} - YAX \right) R^{-T}
$$

$$
B_c = S^{-1} \hat{B}\hat{B}_s^{-1}
$$

$$
C_c = \hat{C}R^{-T}
$$

where $R$ and $S$ are matrices of appropriate dimensions satisfying

$$
RS^T = I - XY.
$$

Proof. Adopting a similar method as in [34], partition-symmetric matrices $P$ and $P^{-1}$ are given as

$$
P = \begin{bmatrix} Y & S \\ S^T & \star \end{bmatrix}, P^{-1} = \begin{bmatrix} X & R \\ R^T & \star \end{bmatrix},
$$

where $X$ and $Y$ are symmetric matrices of dimensions $n \times n$.

Construct the following matrices

$$
\Pi_1 = \begin{bmatrix} X & I \\ R^T & 0 \end{bmatrix}, \Pi_2 = \begin{bmatrix} I & Y \\ 0 & S^T \end{bmatrix}.
$$

Recalling the fact that $PP^{-1} = I$, Eqs. (17) and (18) imply that

$$
P\Pi_1 = \Pi_2
$$

Then, some auxiliary matrices are defined as follows:

$$
\begin{cases}
\hat{A} = SA_cR^T + SB_cC_1X + YB_cR^T + YAX \\
\hat{B} = SB_c\hat{B}_s \\
\hat{C} = C_cR^T
\end{cases}
$$
Based on Eqs. (18)–(20), inequality (14) can be rewritten as

\[
\begin{bmatrix}
-P & 0 & 
\Pi_1^T P \Phi \Pi_1 & 
\Pi_1^T P \bar{E} & 
\Pi_1^T P \bar{B} & 
\Pi_1^T P \bar{M} & 0 \\
* & -I & 
\Phi & 0 & 
0 & 
0 & 0 \\
* & * & 
0 & -\tau \Pi_1^T D^T R_2 & 
0 & 
0 & \epsilon \Pi_1^T \tilde{N}^T \\
* & * & * & -\tau I & 
0 & 
0 & 0 \\
* & * & * & * & -\gamma_s^2 \Xi_\alpha & 
0 & 
0 \\
* & * & * & * & * & -\epsilon I & 0 \\
* & * & * & * & * & * & -\epsilon I \\
\end{bmatrix}
< 0, \quad (21)
\]

where \( \Phi = -\Pi_1^T P \Pi_1 - \tau \Pi_1^T D^T R_1 D \Pi_1 \).

Applying the congruence transformation \( \text{diag} \{ \Pi_1^{-T}, I, \Pi_1^{-T}, I, I, I, I \} \) to Eq. (21) yields the following inequality (22).

\[
\begin{bmatrix}
-P & 0 & 
\bar{P} \bar{\Lambda} & 
PE & 
\bar{P} \bar{B}_2 & 
\bar{P} \bar{M} & 0 \\
* & -I & 
\bar{C} & 0 & 
0 & 
0 & 0 \\
* & * & 
-\tau \Pi_1^T D^T R_1 D & -\tau D^T R_2 & 
0 & 
0 & \epsilon \tilde{N}^T \\
* & * & * & -\tau I & 
0 & 
0 & 0 \\
* & * & * & * & -\gamma_s^2 \Xi_\alpha & 
0 & 
0 \\
* & * & * & * & * & -\epsilon I & 0 \\
* & * & * & * & * & * & -\epsilon I \\
\end{bmatrix}
< 0 
\]

According to the Lemma 1, Eq. (22) is equivalent to

\[
\Sigma + \epsilon^{-1} HH^T + \epsilon G^T G < 0.
\]

where

\[
\Sigma = \begin{bmatrix}
-P & 0 & 
\bar{P} \bar{\Lambda} & 
PE & 
\bar{P} \bar{B}_2 \\
* & -I & 
\bar{C} & 0 & 
0 \\
* & * & 
-\tau \Pi_1^T D^T R_1 D & -\tau D^T R_2 & 
0 \\
* & * & * & -\tau I & 
0 \\
* & * & * & * & -\gamma_s^2 \Xi_\alpha \\
\end{bmatrix},
\]

\[
H = \begin{bmatrix}
\tilde{M}^T & P & 0 & 0 & 0 \\
\end{bmatrix}^T,
\]

\[
G = \begin{bmatrix}
0 & 0 & \tilde{N} & 0 & 0 \\
\end{bmatrix}.
\]

Note that the left hand of Eq. (9) can be constructed as \( \Sigma + HFG + (HFG)^T \). According to Lemma 2, it follows that

\[
\Sigma + HFG + (HFG)^T \leq \Sigma + \epsilon^{-1} HH^T + \epsilon G^T G < 0.
\]

This implies that Eq. (9) is satisfied. Based on Theorem 1, closed-loop system (7) is asymptotically stable, and the prescribed \( H_\infty \) performance index is satisfied. This completes the proof.
3.2. Case B: multiple actuator faults

Consider a class of uncertain nonlinear discrete-time systems subject to multiple actuator faults as follows:

\[
\begin{align*}
    x(k+1) &= (A + \Delta A)x(k) + B(u(k) + \Xi_a(k)f_a(k)) + h(x(k)) \\
    y(k) &= C_1x(k) \\
    z(k) &= C_2x(k)
\end{align*}
\]

where all parameters in Eq. (23) have the same definitions as those in Eq. (1).

Sorting out Eqs. (3) and (23), and using definition (4), a closed-loop system can be derived as

\[
\begin{align*}
    \eta(k+1) &= \bar{A}\eta(k) + \bar{B}\Xi_a(k)f_a(k) + Eh(D\eta(k)) \\
    &= \bar{A}\eta(k) + \bar{B}_1f_a(k) + Eh(D\eta(k)) \\
    z(k) &= \bar{C}\eta(k)
\end{align*}
\]

where

\[
\begin{align*}
    \bar{A} &= \begin{bmatrix} A + \Delta A & BC_c \\ B_cC_1 & A_c \end{bmatrix} = \begin{bmatrix} A & BC_c \\ B_cC_1 & A_c \end{bmatrix} + \begin{bmatrix} M \\ 0 \end{bmatrix}F\begin{bmatrix} N & 0 \end{bmatrix} = \bar{A} + \bar{M}F\bar{N}, \\
    \bar{B} &= \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} B\Xi_a(k) \\ 0 \end{bmatrix}, \quad E = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C_2 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} I & 0 \end{bmatrix}.
\end{align*}
\]

According to Section 2, the control objective is to design a dynamic output feedback controller of form (3) such that

(1) Under the condition \(\Xi_a(k) \equiv 0\), closed-loop system (24) is asymptotically stable; and

(2) with the zero-initial condition, the controlled output satisfies the following performance index:

\[
\sum_{k=0}^{\infty} \mathbb{E}\{||z(k)||^2\} < \gamma_a^2 \sum_{k=0}^{\infty} \mathbb{E}\{||\Xi_a(k)f_a(k)||^2\}
\]

where \(\gamma_a > 0\) is a prescribed scalar.

**Theorem 3.** For a prescribed scalar \(\gamma_a > 0\), if there exists matrix \(P = P^T > 0\) such that

\[
\begin{bmatrix}
    -P & 0 & P\bar{A} & PE & P\bar{B}_2 \\
    * & -I & \bar{C} & 0 & 0 \\
    * & * & -P - \tau D^TR_1D & -\tau D^TR_2 & 0 \\
    * & * & * & -\tau I & 0 \\
    * & * & * & * & -\gamma_a^2\Xi_a
\end{bmatrix} < 0,
\]

holds, where \(\bar{B}_2 = \begin{bmatrix} (B\Xi_a)^T & 0 \end{bmatrix}^T\), then closed-loop system (24) is asymptotically stable, and \(H_\infty\) performance index (25) is satisfied.

Due to the proof of Theorem 3 is similar to that of Theorem 1, it is omitted.

**Theorem 4.** For given scalar \(\gamma_a > 0\), symmetric matrix \(X > 0\), and scalar \(\varepsilon > 0\), if there exist a symmetric matrix \(Y > 0\), real matrices \(\bar{A} \in \mathbb{R}^{n \times n}, \bar{B} \in \mathbb{R}^{n \times p}\), and \(\bar{C} \in \mathbb{R}^{m \times n}\), and a positive scalar \(\tau\)
such that
\[
\begin{bmatrix}
-X & -I & 0 & AX + B\hat{C} & A & I & B\Xi_a & M & 0 \\
* & -Y & 0 & \hat{A} & YA + \hat{B}C_1 & Y & YB\Xi_a & YM & 0 \\
* & * & -I & C_2X & C_2 & 0 & 0 & 0 & 0 \\
* & * & * & -X - \tau XR_1X & -I - \tau XR_1 & -\tau R_1 & 0 & 0 & \varepsilon XN^T \\
* & * & * & * & -Y - \tau R_1 & -\tau R_2 & 0 & 0 & \varepsilon N^T \\
* & * & * & * & * & * & -\varepsilon I & 0 & 0 \\
* & * & * & * & * & * & * & -\varepsilon I & 0 \\
\end{bmatrix} < 0,
\]
holds, then closed-loop system (24) is asymptotically stable and $H_\infty$ performance index (25) can be achieved. Furthermore, the controller parameters are designed as follows:
\[
\begin{aligned}
& A_c = S^{-1}(\hat{A} - \hat{B}C_1X - YB\hat{C} - YAX)R^{-T} \\
& B_c = S^{-1}\hat{B} \\
& C_c = \hat{C}R^{-T}
\end{aligned}
\]
where $R$ and $S$ are matrices of appropriate dimensions satisfying (16).

The proof of Theorem 4 is the same as that of Theorem 2 and is therefore omitted.

### 3.3. Case C: multiple sensor and actuator faults

Consider a class of uncertain nonlinear discrete-time systems subject to multiple sensor and actuator faults simultaneously as described by Eq. (1).

By substituting Eqs. (3) and (4) into Eq. (1), the following closed-loop system is derived.
\[
\begin{aligned}
\eta(k + 1) &= \bar{A}\eta(k) + \bar{B}_1\Xi(k)\bar{f}(k) + Eh(D\eta(k)) \\
z(k) &= \bar{C}\eta(k)
\end{aligned}
\]

where
\[
\bar{B}_1 = \begin{bmatrix} B & 0 \\ 0 & B_c \end{bmatrix}, \quad \Xi(k) = \begin{bmatrix} \Xi_a(k) & 0 \\ 0 & \Xi_a(k) \end{bmatrix}, \quad \bar{f}(k) = \begin{bmatrix} f_a(k) \\ f_s(k) \end{bmatrix}.
\]

Similarly, the control objective is to design a dynamic output feedback controller of form (3) such that
\[
\begin{aligned}
(1) & \quad \text{Under the condition } \Xi(k) \equiv 0, \text{ the closed-loop system (29) is asymptotically stable; and} \\
(2) & \quad \text{under the zero-initial condition, the controlled output satisfies the following performance index:}
\end{aligned}
\]
\[
\sum_{k=0}^{\infty} E\{\|z(k)\|^2\} < \gamma_{\text{both}}^2 \sum_{k=0}^{\infty} E\{\|\Xi(k)\|^2\}.
\]

where $\gamma_{\text{both}} > 0$ is a prescribed scalar.
Theorem 5. For a prescribed scalar $\gamma_{\text{both}} > 0$, if there exists matrix $P = P^T > 0$ such that

$$
\begin{bmatrix}
-P & 0 & \overline{P}A & PE & \overline{P}B_2 \\
* & -I & \overline{C} & 0 & 0 \\
* & * & -P - \tau D^T R_1 D & -\tau D^T R_2 & 0 \\
* & * & * & -\tau I & 0 \\
* & * & * & * & -\gamma_{\text{both}}^2 \Xi
\end{bmatrix} < 0.
$$

(31)

holds, where $\overline{B}_2 = B_1 \Xi$, and $\Xi = \text{diag}\{\Xi_a, \Xi_s\}$, then the closed-loop system (29) is asymptotically stable, and $H_\infty$ performance index (30) is satisfied. The proof is omitted for saving space.

Theorem 6. For given scalar $\gamma_{\text{both}} > 0$, symmetric matrix $X > 0$, and scalar $\epsilon > 0$, if there exist symmetric matrix $Y > 0$, real matrices $\hat{A} \in \mathbb{R}^{n \times n}$, $\hat{B} \in \mathbb{R}^{n \times p}$, and $\hat{C} \in \mathbb{R}^{m \times n}$, and a positive scalar $\tau$ such that

$$
\begin{bmatrix}
-X & -I & 0 & AX + B\hat{C} & A & I & B\Xi_a & 0 & M & 0 \\
* & -Y & 0 & \hat{A} & YA + \hat{B}\Xi_a^{-1} C_1 & Y & YB\Xi_a & \hat{B} & YM & 0 \\
* & * & -I & C_2X & C_2 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & -X - \epsilon XR_1 X & -I - \epsilon XR_1 & -\tau X R_2 & 0 & 0 & 0 & \epsilon X N^T \\
* & * & * & * & -Y - \tau R_1 & -\tau R_2 & 0 & 0 & 0 & \epsilon N^T \\
* & * & * & * & * & -\tau I & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -\gamma_{\text{both}}^2 \Xi_a & 0 & 0 & 0 \\
* & * & * & * & * & * & * & -\gamma_{\text{both}}^2 \Xi_s & 0 & 0 \\
* & * & * & * & * & * & * & * & -\epsilon I & 0 \\
* & * & * & * & * & * & * & * & -\epsilon I & \epsilon I
\end{bmatrix} < 0
$$

(32)

holds, then the closed-loop system (29) is asymptotically stable, and $H_\infty$ performance index (30) is satisfied. Furthermore, the controller parameters are designed as Eq. (15).

The proof is omitted for simplicity.

Remark 3. This study is an extension of the up-to-date publication [29], which also studies the reliable $H_\infty$ control problem for discrete-time system with IFs. In [29], there is an assumption that all sensor or actuator faults must occur simultaneously, which is not reasonable. By employing two sets of independent stochastic variables, this study proposes a more reasonable and general fault model, in which the sensor or actuator faults could occur separately.

Remark 4. In practical systems, IFs occur on components, such as sensors and actuators. Usually, the dynamic performances of IFs are different, which depend on the IFs themselves and the circumstances. Under some situations, the IFs would deteriorate and even evolve into PFs while the IFs occur very frequently with the time going. In this study, it is assumed that the IFs are square summable such that the problem can be solved in the framework of reliable $H_\infty$ control. In future study, the dynamic properties of IFs will be paid more attention to, and more practical and precise mathematical descriptions (such as by means of Markov chain) of IFs will be proposed and investigated.
4. Simulation examples

In this section, two simulation examples are introduced to verify the effectiveness of the proposed control strategy.

Example 1. Consider an uncertain nonlinear discrete-time system with the following system matrices:

$$
A = \begin{bmatrix}
-0.3 & -0.2 & -0.3 \\
0.1 & -1.1 & 0.1 \\
0.3 & 0.2 & -0.2
\end{bmatrix},
B = \begin{bmatrix}
0.4 & 0.3 \\
0.1 & 0.5 \\
0.2 & 0.1
\end{bmatrix},
C_1 = \begin{bmatrix}
0 & 0.1 & -0.2 \\
0.1 & -0.1 & 0.4
\end{bmatrix},
C_2 = \begin{bmatrix}
0.3 & 0.5 & 0.7
\end{bmatrix}.
$$

The nonlinear function is assumed as

$$
h(x(k)) = \begin{bmatrix}
0.02x_1(k) \sin^2(x_1(k)) - 0.01(x_1(k) - x_2(k) + x_3(k)) \\
-0.01(x_1(k) - x_3(k)) \\
-0.01(x_2(k) + x_3(k))
\end{bmatrix},
$$

which can be bounded by

$$
T_1 = \begin{bmatrix}
0.01 & 0.01 & -0.01 \\
0.01 & 0.02 & 0.04 \\
-0.02 & 0.01 & 0.02
\end{bmatrix},
T_2 = \begin{bmatrix}
-0.01 & 0.01 & -0.01 \\
-0.03 & -0.02 & -0.02 \\
0.02 & -0.03 & -0.04
\end{bmatrix}.
$$

In the simulation, the probability of occurrence of a sensor and actuator fault is assumed to be $\pi_1 = \tau_1 = 0.02$, $\pi_2 = \tau_2 = 0.03$, and $f_s(k) = f_a(k) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$. Let $F = \sin(k)$, and the prescribed $H_\infty$ performance level $\gamma_s = \gamma_a = \gamma_{both} = 0.9$. To solve matrix inequalities Eqs. (14), (27) and (32), some related parameters are chosen as $X = \text{diag}[0.6, 0.6, 0.6], \epsilon = 0.1, M = [0.03 \ 0.03 \ 0.03]^T, N = [0.1 \ 0.1 \ 0.1]$.

Solving matrix inequalities Eqs. (14), (27), and (32), respectively, one can obtain the matrices for the dynamic output-feedback controller as below.

For Case A

$$
A_c = \begin{bmatrix}
-0.2694 & -0.2324 & -0.0947 \\
0.1070 & 0.1539 & 0.0326 \\
0.5038 & -1.2720 & 0.3233
\end{bmatrix},
B_c = \begin{bmatrix}
-0.1053 & 0.1471 \\
-0.1129 & 0.1135 \\
6.7077 & -3.9234
\end{bmatrix},
$$

$$
C_c = \begin{bmatrix}
-0.1080 & -0.3451 & 0.3869 \\
0.4542 & -0.2142 & -0.5639
\end{bmatrix}.
$$

For Case B

$$
A_c = \begin{bmatrix}
-0.1876 & 0.1182 & 0.0945 \\
-0.2475 & 0.4720 & -0.2399 \\
0.5622 & -0.1825 & 0.8125
\end{bmatrix},
B_c = \begin{bmatrix}
8.4029 & 5.0548 \\
-12.9807 & -4.6449 \\
22.2821 & 10.4741
\end{bmatrix},
$$

$$
C_c = \begin{bmatrix}
0.0336 & -1.0417 & 0.4521 \\
0.1735 & 0.9528 & -1.1084
\end{bmatrix}.
For Case C

\[
A_c = \begin{bmatrix}
-0.2591 & -0.2487 & -0.1107 \\
0.0606 & 0.1644 & 0.0228 \\
0.6565 & -1.3604 & 0.3751
\end{bmatrix},
B_c = \begin{bmatrix}
-0.1636 & 0.1862 \\
-0.2254 & 0.2085 \\
7.9145 & -4.0537
\end{bmatrix},
\]

\[
C_c = \begin{bmatrix}
-0.0875 & -0.3755 & 0.4015 \\
0.4977 & -0.1431 & -0.5757
\end{bmatrix}.
\]

The simulation results of Example 1 are shown in Figs. 2–8. The state trajectories of the uncontrolled system are given in Fig. 2, which demonstrates that the system is unstable. Figs. 3, 4...
5 and 7 show the state trajectories of the controlled system for Cases A, B, and C, respectively. It is clear that the reliable controllers can maintain the closed-loop system in a stable state and make the system converge rapidly to zero after the advent of multiple sensor and/or actuator faults. The designed controller signals of Cases A, B, and C are shown in Figs. 4, 6, and 8, respectively.
Example 2. In this part, the proposed PFTC strategy is applied to an aircraft engine system. The original model is a linearized continuous-time system [35]. With a sampling period of 0.1 s, one can obtain a discrete-time model, where its parameters are as following:

$$A = \begin{bmatrix} 0.7225 & 1.0113 \\ 1.1317 & 0.6958 \end{bmatrix}, B = \begin{bmatrix} 0.2315 & 0.3364 \\ 0.2678 & 0.4070 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0.0056 & 0 & -0.1190 \\ 0 & 1 & -0.0021 & 0.0003 & 0.0858 \end{bmatrix}^T.$$
It is assumed that when modeling the aircraft engine system, there exist norm bounded parameter uncertainties and sector-bounded nonlinear disturbance. Therefore, apart from the system parameters $A$, $B$, and $C_1$, other parameters are set as follows:

\[ C_2 = \begin{bmatrix} 0.1 & 0.1 \end{bmatrix}, \]
\[ M = \begin{bmatrix} 0.03 & 0.03 \end{bmatrix}^T,\ N = \begin{bmatrix} 0.01 & 0.01 \end{bmatrix}, \]
\[ T_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.2 \end{bmatrix},\ T_2 = \begin{bmatrix} -0.1 & 0 \\ -0.1 & 0.1 \end{bmatrix}, \]
\[ \epsilon = 0.1,\ X = \text{diag}(0.5, 0.5), \]
\[ \Xi_{\alpha} = \text{diag}(0.02, 0.04), \]
In addition, the sector-bounded nonlinear function is of the form

$$h(x(k)) = \frac{1}{2} \begin{bmatrix} 0.3x_{1k} + 0.1x_{2k} + 0.1x_{1k} + 0.3x_{1k} + 0.3x_{2k} \end{bmatrix}^T.$$  

In this simulation, only Case C is considered for simplicity, and the amplitude of the sensor and actuator in each channel is assumed to be 1. Let $F = \sin(k)$, and the prescribed $H_\infty$ performance level is taken as $\gamma_{both} = 9$. 

Fig. 10. State trajectories of the closed-loop aircraft engine system with intermittent faults.

Fig. 11. Controller signal of the closed-loop aircraft engine system with intermittent faults.
Solving LMI (32), one can obtain the matrices for the dynamic output-feedback controller as below.

\[
A_c = \begin{bmatrix}
0.2485 & -0.2951 \\
-0.1900 & 1.8635
\end{bmatrix},
B_c = \begin{bmatrix}
0.8454 & 0.0282 & 0 & 0 & -0.0273 \\
2.9979 & 3.0116 & 0 & 0.0018 & 0.1517
\end{bmatrix},
C_c = \begin{bmatrix}
-15.8184 & -3.1511 \\
10.4989 & 0.5439
\end{bmatrix}.
\]

The simulation results of Example 2 are shown in Figs. 9–11, which demonstrate the effectiveness of the proposed strategy.

5. Conclusions

In this study, a PFTC strategy was investigated for a class of uncertain nonlinear discrete-time systems subject to multiple IFs, where the multiple IFs were depicted by a set of Bernoulli-distributed variables. In the framework of reliable \(H_\infty\) control, sensor IFs and/or actuator IFs are studied, and three dynamic output-feedback controllers were designed such that the closed-loop systems maintain asymptotical stability and satisfy a prescribed \(H_\infty\) performance index. The provided two simulation examples demonstrate the effectiveness of the proposed FTC method.

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References