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Brief paper

On almost sure and mean square convergence of P-type ILC under randomly varying iteration lengths☆

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ABSTRACT

This note proposes convergence analysis of iterative learning control (ILC) for discrete-time linear systems with randomly varying iteration lengths. No prior information is required on the probability distribution of randomly varying iteration lengths. The conventional P-type update law is adopted with Arimoto-like gain and/or causal gain. The convergence both in almost sure and mean square senses is proved by direct math calculating. Numerical simulations verifies the theoretical analysis.

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1. Introduction

Repeatability may be a basic requirement for learning, because a learning algorithm usually works by trying and correcting. Meanwhile, many practical systems perform the same task again and again and thus satisfy the requirement of learning. For example, a chemical process is operated repeatedly in a plant to obtain batches of product. For this kind of systems, the information generated in the previous iterations could be used to improve the quality of products. This is the starting point of iterative learning control (ILC), aiming to improve the system performance by learning (Ahn, Chen, & Moore, 2007; Bristow, Tharayil, & Alleyne, 2006; Shen & Wang, 2014; Wang, Gao, & Doyle III, 2009). As a matter of fact, ILC updates its control signal mainly using information of previous iterations rather than only the information of previous time instances in current iteration. Besides, it has simple control structure but effective capacity, thus has been applied to a lot of practical systems such as high-speed rail train (Sun, Hou, & Li, 2013), permanent magnet step motors (Bifaretti, Tomei, & Verrelli, 2011), robotic-assisted biomedical system (Xu, Chu, & Rogers, 2014), etc.

In order to gain a good performance, it is usually required that the tracking task of ILC repeats in a fixed time interval. However, in many practical applications, the trail length may vary among different iterations. Two biomedical systems, which are functional electrical stimulation for upper limb movement and for gait assistance, respectively, were given in Seel, Schauer, and Raisch (2011) to show this case. Subjected to complex factors and unknown dynamics, the learning process may end before the whole stimulation profile is completed. Another example was provided in Longman and Mombaur (2006), where humanoid robot was considered to follow some gait by ILC and repetitive control. One of the most important issues is that durations of the phases are not the same from iteration to iteration during the learning process. Therefore, the repeatability of iteration length is no longer valid in these cases. This motivates us to make efforts on the ILC under randomly varying iteration lengths.

Some studies have been made on this topic. In Seel et al. (2011), the maximum of iteration length was defined as full-length, and then the tracking error trajectory of all the non-full-length learning iterations were extended to be of full-length by appending zero elements at the lost positions. ILC algorithm was designed based on the modified tracking error trajectory to fulfill the requirements of standard ILC. It is shown that the algorithm could improve the first input samples sequentially, while maintaining the performance of last samples until long enough iteration occurs. In Li, Xu, and

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Huang (2014), the iteration length was assumed to be randomly varying and the iteration average operator technique was used to design ILC algorithms. It was proved that the mathematical expectation of tracking error converges to zero with the help of λ-norm technique. The varying iteration model there is that the actual iteration length could be either longer or shorter than the desired iteration length. If the actual iteration length exceeds the desired one, then the redundant points of tracking error are cut off, while on the contrary the absent points of tracking error are set as zero. The nonlinear and continuous-time case of Li et al. (2014) was given in Li, Xu, and Huang (2015) following similar techniques. In short, more efforts on the analysis and performance in probability sense should be made when iteration length varies randomly.

In this note, we build strong convergence property of ILC algorithms for linear system with randomly varying iteration lengths. It should be emphasized that the important significance is not to propose an alternative algorithm of previous results, but to reveal that the conventional P-type ILC algorithm retains robustness and strong convergence against randomly varying iteration lengths.

There are several major differences between this note and (Li et al., 2014; Seel et al., 2011) from the perspectives of convergence analysis technique and convergence property. In the first place, the analysis of Seel et al. (2011) was directly made according to tracking error of adjacent iterations following a determinate way. In Li et al. (2014), the random variable describing the varying iteration length was turned into a deterministic value by taking mathematical expectation and then no randomness had to be considered in the following derivations. Thus both Seel et al. (2011) and Li et al. (2014) showed the convergence following the determinate way. In contrast, this note proposes a novel analysis technique from the perspective of probability theory. Moreover, the convergence property of Li et al. (2014) and Seel et al. (2011) are much weaker than this note. In Seel et al. (2011), it proved that the tracking error was monotonically decreasing in 1-norm and ∞-norm senses but the limitation of the tracking error might not be zero. In Li et al. (2014), the expectation of tracking error was shown convergent to zero. However, this does not imply that the actual tracking error was small enough since the variance could be very large. Unlike Li et al. (2014) and Seel et al. (2011), the almost sure and mean square convergence results proposed in this note are the strongest convergence in probability theory. Last but by no means least, both Seel et al. (2011) and this note require no prior information on the probability distribution of random varying lengths, while the latter information was used in Li et al. (2014) for the design of learning matrix.

Specifically speaking, the contributions of this note are listed as follows: (a) the conventional P-type ILC update law is used and shown effective and robust under randomly varying iteration lengths; (b) both almost sure and mean square convergence of the input sequence is proved by direct calculations; (c) no prior information on the probability distribution of randomly varying iteration lengths is required, thus is more suitable for practical applications.

The rest of the note is arranged as follows. Section 2 gives problem formulation, where the randomly varying iteration lengths is defined. In Section 3, the conventional ILC algorithm is used with minor modifications on tracking error. Strict convergence analysis both in almost sure sense and mean square sense is proposed in Section 4. Section 5 shows an illustration simulation and Section 6 concludes this note.

Notations. $\mathbb{E}$ denotes the mathematical expectation. $\lambda(A)$ is the eigenvalue of a matrix A, while $\rho(A)$ is the spectral radius. $\|A\|$ is an induced norm of a matrix A. The subscript T denotes the transpose, and $\otimes$ denotes the Kronecker product. $P(\cdot)$ denotes the probability of an event. $\mathbb{E}X$ denotes the mathematical expectation of a random variable X. The abbreviation "i.o." is short for "infinitely often".

2. Problem formulation

Consider the following linear time-varying system

$$\begin{aligned} x(t+1,k) &= A_t x(t,k) + B_t u(t,k) \\ y(t,k) &= C_t x(t,k) \end{aligned} \tag{1}$$

where $x(t,k) \in \mathbb{R}^n$, $u(t,k) \in \mathbb{R}^p$, and $y(t,k) \in \mathbb{R}^q$ denote state, input, and output, respectively. $k = 0, 1, \ldots$ and $t = 0, 1, \ldots, N$ denote the iteration index and discrete time index, respectively, and N is the maximum of iteration lengths. A_t, B_t, and C_t are system matrices with appropriate dimensions. It is assumed that $C_{t+1}B_t$ is of full-column-rank, which means the relative degree is 1.

If the operation length of all iterations is identical, i.e., N, then the system model (1) could be lifted as follows

$$Y_k = HU_k + Y_{k0} \tag{2}$$

where

$$H = \begin{bmatrix} C_1B_0 & 0 & \cdots & 0 \\ C_2A_1B_0 & C_2B_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_NA_{N-1}\cdots A_1B_0 & \cdots & \cdots & C_NB_{N-1} \end{bmatrix} \tag{3}$$

and

$$Y_k = \left[y^T(1,k), \ldots, y^T(N,k)\right]^T$$

$$U_k = \left[u^T(0,k), \ldots, u^T(N-1,k)\right]^T$$

$$Y_{k0} = \left[(C_1A_0)^T, \ldots, (C_NA_{N-1}\cdots A_0)^T\right]^T x(0,k).$$

Let $y(t,d)$, $t = 0, 1, \ldots, N$ be the desired trajectory. Assume that, for a realizable trajectory $y(t,d)$, there is a unique control input $u(t,d)$ such that

$$Y_d = HU_d + Y_{d0} \tag{4}$$

where Y_d, U_d, and Y_{d0} are defined similar to Y_k, U_k, and Y_{k0}, respectively, by replacing $y(t,k)$, $u(t,k)$, and $x(0,k)$ with $y(t,d)$, $u(t,d)$, and $x(0,d)$. For expression clarity and making our idea clear, it is assumed that the identical initial condition holds, i.e., $Y_{k0} = Y_{d0}$.

Then the control objective is to design ILC algorithm such that $Y_k \to Y_d$ as the iteration index k approaches to infinity if the iteration length is fixed as N. However, the actual iteration length may vary from iteration to iteration randomly. Besides, it is rational to have a lower bound of actual iteration lengths, named by $\overline{N}$, $\overline{N} < N$, such that the actual iteration length varies in $\{\overline{N}, \overline{N}+1, \ldots, N\}$ randomly. In other words, for the first $\overline{N}$ time instances one could always get the actual output, while for the left time instances the system output occurs randomly. Thus there are $N - \overline{N} + 1$ possible output trajectories. For the kth iteration, it ends at the N_k-th time instance, that is, only the first N_k outputs are received, where $\overline{N} \le N_k \le N$. In order to make it clear, a cutting-operator $\lfloor \cdot \rfloor_{N_k}$ is introduced to Y_k which means the last $N - N_k$ outputs of Y_k are removed, i.e., $\lfloor \cdot \rfloor_{N_k} : \mathbb{R}^{Nq} \to \mathbb{R}^{N_k q}$. Therefore, the control objective of this note is to design ILC update law such that $\lfloor Y_k \rfloor_{N_k} \to \lfloor Y_d \rfloor_{N_k}$ as k approaches to infinity. For notations concise, let $m = N - \overline{N} + 1$ in the followings.

As one can see now, there are $N - \overline{N} + 1$ possible iteration lengths which occur randomly. In order to cope with these different scenarios, let the probability that the iteration length is of $\overline{N}$, $\overline{N}+1, \ldots, N$ be $p_1, p_2, \ldots, p_m$, respectively. That is, $P(\mathcal{A}_{\overline{N}}) = p_1$, $P(\mathcal{A}_{\overline{N}+1}) = p_2, \ldots, P(\mathcal{A}_N) = p_m$, $\forall k$, where $\mathcal{A}_l$ denotes the event that the iteration length is l, i.e., $\lfloor Y_k \rfloor_{N_k} = \lfloor Y_k \rfloor_l$, $\overline{N} \le l \le N$. Obviously, $p_i > 0$, $1 \le i \le m$, and

$$p_1 + p_2 + \cdots + p_m = 1. \tag{5}$$

It should be pointed out that no probability distribution is assumed on p_i, thus the above expression on randomly varying iteration length is general.

Remark 1. In Li et al. (2014), the authors introduced a sequence of random variables satisfying Bernoulli distribution based on $p_1, \ldots, p_m$ to model the probability of the occurrence of the last $N - \overline{N}$ outputs. Then the random variable was multiplied to the corresponding tracking error as a modified tracking error. The analysis objective was therefore transformed to show that the mathematical expectation of these $N - \overline{N}$ modified tracking error converges to zero. While in this note, we prove the convergence both in almost sure sense and mean square sense straightforward according to the iteration length probabilities $p_1, \ldots, p_m$ by direct calculations. Besides, it is worth pointing out that these probabilities are only used for the analysis, while the design condition of learning algorithm requires none prior information on these probabilities, which is more suitable for practical applications.

3. ILC design

Notice that the iteration length cannot exceed the maximum length, thus only two cases of tracking error need to be considered. If the iteration length equals the maximum length, then the tracking error is a normal one with dimension Nq; while if the iteration length is shorter than the maximum length, then the tracking error at the absent time instances are missing, which therefore could not be used for input update. For the latter case, we could append zeros to the absent time instances so that the tracking error is again transformed to be a normal one with dimension Nq. In other words, when the kth actual output length is not up to the maximum, namely, $N_k < N$, then the tracking errors are defined as

$$e(t,k) = \begin{cases} y(t,d) - y(t,k), & 1 \le t \le N_k \\ 0, & N_k < t \le N. \end{cases} \tag{6}$$

Denote

$$E_k = \left[e^T(1,k), \ldots, e^T(N,k)\right]^T. \tag{7}$$

The control update law is thus defined as follows

$$U_{k+1} = U_k + LE_k \tag{8}$$

where L is a learning gain matrix to be designed later.

Noting that $E_k \neq Y_d - Y_k$ if $N_k < N$, we have to fill the gap by introducing the following matrix

$$M_{N_k} = \begin{bmatrix} I_{N_k} \otimes I_q & 0 \\ 0 & \mathbf{0}_{(N-N_k)} \otimes I_q \end{bmatrix}, \quad \overline{N} \le N_k \le N \tag{9}$$

where I_l and $\mathbf{0}_l$ denote unit matrix and zero matrix with dimension of $l \times l$, respectively.

Then one has

$$E_k = M_{N_k}(Y_d - Y_k) = M_{N_k}H(U_d - U_k).$$

Now (8) leads to

$$\begin{aligned} U_{k+1} =& U_k + LE_k \\ =& U_k + LM_{N_k}H(U_d - U_k). \end{aligned}$$

Subtracted both sides of last equation from U_d, one has

$$\begin{aligned} U_d - U_{k+1} =& U_d - U_k - LM_{N_k}H(U_d - U_k) \\ =& (I - LM_{N_k}H)(U_d - U_k). \end{aligned}$$

That is,

$$\Delta U_{k+1} = (I - LM_{N_k}H)\Delta U_k \tag{10}$$

where $\Delta U_k \triangleq U_d - U_k$.

Notice that N_k is a random variable valued from $\{\overline{N}, \ldots, N\}$, therefore M_{N_k} is a random matrix, which further results in that $I - LM_{N_k}H$ is a random matrix. Thus one could introduce m binary random variables γ_i, $1 \le i \le m$ such that $\gamma_i \in \{0, 1\}$,

$$\gamma_1 + \gamma_2 + \cdots + \gamma_m = 1$$

and

$$P(\gamma_i = 1) = P(\mathcal{A}_{\overline{N}-1+i}) = p_i, \quad 1 \le i \le m.$$

Then (10) could be reformulated as

$$\begin{aligned} \Delta U_{k+1} =& [\gamma_1(I - LM_{\overline{N}}H) + \gamma_2(I - LM_{\overline{N}+1}H) \\ & + \cdots + \gamma_m(I - LM_N H)]\Delta U_k. \end{aligned} \tag{11}$$

Consequently, the zero-convergence of the original update law (10) could be achieved by analyzing zero-convergence of (11).

Denote $\Gamma_i = I - LM_{\overline{N}-1+i}H$, $1 \le i \le m$. Then (11) could be simplified as

$$\Delta U_{k+1} = (\gamma_1\Gamma_1 + \gamma_2\Gamma_2 + \cdots + \gamma_m\Gamma_m)\Delta U_k. \tag{12}$$

It is easy to see that all γ_i are dependent, since whenever one of them values 1, then all the others have to value 0.

In order to give the design condition of learning gain matrix L, we first calculate the mean and covariance along the sample path.

Let $\mathcal{S} = \{\Gamma_i, 1 \le i \le m\}$, and denote

$$Z_k = X_k X_{k-1} \cdots X_1 X_0 \tag{13}$$

where X_k is a random matrix taking values in $\mathcal{S}$ with $P(X_k = \Gamma_i) = P(\gamma_i = 1) = p_i$, $1 \le i \le m$, $\forall k$. Now we have the following equation from (12)

$$\Delta U_{k+1} = Z_k \Delta U_0. \tag{14}$$

The following two lemmas are given for further analysis.

Lemma 1. *Let $\mathcal{S}^k = \{Z_k$: taken over all sample paths$\}$, then the mean of the $\mathcal{S}$, denoted by K_k, is given recursively by*

$$K_k = \left(\sum_{i=1}^m p_i\Gamma_i\right)K_{k-1}. \tag{15}$$

Proof. Let $\mathcal{S}_i^k = \{Z_k \in \mathcal{S}^k : X_k = \Gamma_i\}$, $i = 1, 2, \ldots, m$. It is obvious that $\mathcal{S}^k$ is the disjoint union of $\mathcal{S}_i^k$.

Let

$$K_k = \sum_{Z_k \in \mathcal{S}^k} P\{Z_k\}Z_k. \tag{16}$$

By the independence of X_j, one can decompose the above sum

$$\begin{aligned} K_k =& \sum_{Z_k \in \mathcal{S}^k} P\{Z_k\}Z_k \\ =& \sum_{Z_{k-1} \in \mathcal{S}^{k-1}} \sum_{i=1}^m P\{X_k = \Gamma_i\}P\{Z_{k-1}\}\Gamma_i Z_{k-1} \\ =& \sum_{i=1}^m P\{\gamma_i = 1\}\Gamma_i \sum_{Z_{k-1} \in \mathcal{S}^{k-1}} P\{Z_{k-1}\}Z_{k-1} \\ =& \sum_{i=1}^m p_i\Gamma_i K_{k-1} = \left(\sum_{i=1}^m p_i\Gamma_i\right)K_{k-1}. \end{aligned}$$

Thus the proof is completed. □

Lemma 2. *Let $\mathcal{S}^k = \{Z_k$: taken over all sample paths$\}$, then the covariance of the $\mathcal{S}$, denoted by V_k, is given by*

$$V_k = F_k - K_k K_k^T \tag{17}$$

where F_k is generated recursively as

$$F_k = \sum_{i=1}^m p_i\Gamma_i F_{k-1}\Gamma_i^T. \tag{18}$$

Proof. The covariance is calculated as

$$V_k = \sum_{Z_k \in \mathcal{S}^k} P\{Z_k\}(Z_k - K_k)(Z_k - K_k)^T.$$

Then by decomposition it leads to the following derivation

$$\begin{aligned}
V_k =& \sum_{Z_{k-1}\in\mathcal{S}^{k-1}} \sum_{i=1}^m \Big[P\{X_k = \Gamma_i\} P\{Z_{k-1}\} \\
&\times (\Gamma_i Z_{k-1} - K_k)(\Gamma_i Z_{k-1} - K_k)^T \Big] \\
=& \sum_{Z_{k-1}\in\mathcal{S}^{k-1}} \sum_{i=1}^m \Big[P\{\gamma_i = 1\} P\{Z_{k-1}\} \\
&\times (\Gamma_i Z_{k-1} - K_k)(\Gamma_i Z_{k-1} - K_k)^T \Big] \\
=& \sum_{i=1}^m p_i \Big[\sum_{Z_{k-1}\in\mathcal{S}^{k-1}} P\{Z_{k-1}\}\Gamma_i Z_{k-1} Z_{k-1}^T \Gamma_i^T \\
&- \sum_{Z_{k-1}\in\mathcal{S}^{k-1}} P\{Z_{k-1}\} K_k Z_{k-1}^T \Gamma_i^T \\
&- \sum_{Z_{k-1}\in\mathcal{S}^{k-1}} P\{Z_{k-1}\}\Gamma_i Z_{k-1} K_k^T + \sum_{Z_{k-1}\in\mathcal{S}^{k-1}} P\{Z_{k-1}\} K_k K_k^T \Big] \\
=& \sum_{i=1}^m p_i \Big[\sum_{Z_{k-1}\in\mathcal{S}^{k-1}} P\{Z_{k-1}\}\Gamma_i Z_{k-1} Z_{k-1}^T \Gamma_i^T \\
&- K_k K_{k-1}^T \Gamma_i^T - \Gamma_i K_{k-1} K_k^T + K_k K_k^T \Big].
\end{aligned}$$

From Lemma 1 it is noticed that

$$\sum_{i=1}^m p_i K_k K_{k-1}^T \Gamma_i^T = K_k K_k^T$$

$$\sum_{i=1}^m p_i \Gamma_i K_{k-1} K_k^T = K_k K_k^T.$$

Thus one has

$$V_k = \sum_{i=1}^m p_i \Gamma_i \left(\sum_{Z_{k-1}\in\mathcal{S}^{k-1}} P\{Z_{k-1}\} Z_{k-1} Z_{k-1}^T \right) \Gamma_i^T - K_k K_k^T.$$

On the other hand

$$\begin{aligned}
V_k =& \mathbb{E}(Z_k - K_k)(Z_k - K_k)^T \\
=& \mathbb{E} Z_k Z_k^T - K_k K_k^T \\
=& \sum_{Z_k\in\mathcal{S}^k} P\{Z_k\} Z_k Z_k^T - K_k K_k^T.
\end{aligned}$$

Let $F_k = \sum_{Z_k\in\mathcal{S}^k} P\{Z_k\} Z_k Z_k^T$, then it is obvious that

$$F_k = \sum_{i=1}^m p_i \Gamma_i F_{k-1} \Gamma_k^T$$

by combining the last two expressions of V_k. This completes the proof. □

Now return to the iterative equation (12) and design the learning gain matrix L. Notice that H is a block lower triangular matrix, and M_i is a block diagonal matrix, $\overline{N} \le i \le N$. Thus there is a large degree of freedom on the design of learning gain L. As a matter of fact, L could be partitioned as $L = [L_{i,j}]$, $1 \le i, j \le N$, where $L_{i,j}$ is a submatrix of $p \times q$. Two types of L are designed as follows.

- *Arimoto-like gain*: The diagonal blocks of L, i.e., $L_{i,i}$, $1 \le i \le N$, are valued, while the other blocks are set 0.
- *Causal gain*: The blocks in the lower triangular part of L, i.e., $L_{i,j}$, $i \ge j$, are valued, while the other blocks are set 0.

No matter which type of L mentioned above is adopted, it is easy to find that the coupled matrix $LM_{N_K}H$ is still a block lower triangular matrix whose diagonal blocks are $L_{t,t}C_tB_{t-1}$, $1 \le t \le N_k$ or 0, $N_k + 1 \le t \le N$. Thus one could simply design $L_{t,t}$, satisfying

$$0 < I - L_{t,t}C_tB_{t-1} < I. \tag{19}$$

Remark 2. If Arimoto-like gain is selected, one could find that the update law (8) could be formulated on each time instance as $u(t, k+1) = u(t, k) + L_{t+1,t+1}e(t+1, k)$, which further reduces the computational burden brought by high-dimension of the lifted model (2). However, on the other hand, the causal gain may offer more flexibilities for us, since no condition is required on the block $L_{i,j}$, $i > j$.

Remark 3. Different from (19), the condition given in Li et al. (2014) is that $\sup \|I - p(t)LCB\| \le \theta$ where $0 \le \theta < 1$ and $p(t)$ is the occurrence probability of output at t. Thus it is required in Li et al. (2014) that the probability of each iteration length is known prior. In this paper, the requirement of L only depends on the input–output coupling matrix CB and thus no prior information on probabilities of randomly varying iteration length is needed, which therefore is more suitable for implementation.

4. Strong convergence properties

Based on Lemma 1, the following theorem establishes the convergence in mathematical expectation sense.

Theorem 1. *Consider system (2) with randomly varying iteration length and use control update law (8). The mathematical expectation of tracking error E_k, i.e., $\mathbb{E}E_k$, converges to zero if the learning gain L satisfies (19).*

Proof. By (14) one has

$$\mathbb{E}\Delta U_{k+1} = K_k \mathbb{E}\Delta U_0.$$

Then by recurrence of the mean K_k, i.e., (15), it is obvious to have

$$\mathbb{E}\Delta U_k = \left(\sum_{i=1}^m p_i \Gamma_i \right)^k \mathbb{E}\Delta U_0.$$

Thus it is sufficient to show that $\rho\left(\sum_{i=1}^m p_i\Gamma_i\right) < 1$. By (19) one can have the consequence that each eigenvalue of $I - LM_{N_K}H$, denoting as $\lambda_j(I - LM_{N_K}H)$, satisfies

$$0 < \lambda_j(I - LM_{N_K}H) \le 1$$

$1 \le j \le Np$, $\overline{N} \le N_k \le N$. An eigenvalue $\lambda_j(I - LM_{N_K}H)$ equals 1 if and only if there are some outputs missed, i.e., $N_k < N$.

Note that Γ_i is a block lower triangular matrix, thus the eigenvalues actually are collection of eigenvalues of all the diagonal blocks. Take the lth diagonal blocks from top to bottom of each alternative matrix Γ_i into account. For the case $1 \le l \le \overline{N}$, it is observed that all the eigenvalues of the lth diagonal block are positive and less than 1. While for the case $\overline{N} + 1 \le l \le N$, all the eigenvalues of the lth diagonal block are positive and not larger than 1. Meanwhile, not all eigenvalues of the lth diagonal blocks are equal to 1 because of the existence of $\Gamma_m = I - LM_NH = I - LH$, whose eigenvalues are all less than 1, $\overline{N}+1 \le l \le N$. By noticing that $\sum_{i=1}^{m} p_i = 1$, it is obvious that, for any $1 \le j \le Np$,

$$0 < \sum_{i=1}^{m} p_i \lambda_j(\Gamma_i) < 1. \tag{20}$$

Noting that $\mathbb{E}E_k = \mathbb{E}\lfloor H\Delta U_k \rfloor_{N_k}$, the proof is completed. □

Remark 4. Theorem 1 presents the convergence property in mathematical expectation sense. In other words, the expectation of tracking error converges to zero for the conventional P-type law (8). This kind of convergence is also obtained in Li et al. (2014), where an iteration-average operator is introduced to cope with the randomness of iteration length. However, we take a different analytical approach.

The following theorem shows the almost sure convergence property.

Theorem 2. *Consider system (2) with randomly varying iteration length and use control update law (8). The tracking error E_k converges to zero almost surely if the learning gain L satisfies (19).*

Proof. Concerning (20), for the 2-norm $\|\cdot\|$, one has that

$$0 < \sum_{i=1}^{m} p_i \|\Gamma_i\| < 1. \tag{21}$$

Thus one can find a constant $0 < \delta < 1$ such that

$$0 < \sum_{i=1}^{m} p_i \|\Gamma_i\| < \delta \tag{22}$$

since the number m is limited, denoting the number of different possible formulation of Γ_i and thus the above is a definite summation.

Noticing that the iteration length vary independently from iteration to iteration and N_k possesses identical distribution of k, it follows from (10), (11), and (12) that

$$\begin{aligned}\mathbb{E}\|\Delta U_k\| =& \mathbb{E}\|Z_{k-1}\|\mathbb{E}\|\Delta U_0\| \\ =& \mathbb{E}\|X_{k-1}\cdots X_1 X_0\|\mathbb{E}\|\Delta U_0\| \\ =& \mathbb{E}\|X_{k-1}\|\mathbb{E}\|X_{k-2}\|\cdots \mathbb{E}\|X_0\|\mathbb{E}\|\Delta U_0\| \\ =& (\mathbb{E}\|X_{k-1}\|)^k\, \mathbb{E}\|\Delta U_0\|.\end{aligned}$$

On the other hand

$$\begin{aligned}\mathbb{E}\|X_{k-1}\| =& \mathbb{E}\|\gamma_1\Gamma_1 + \gamma_2\Gamma_2 + \cdots + \gamma_m\Gamma_m\| \\ =& \sum_{i=1}^{m} P(\gamma_i = 1)\|\gamma_1\Gamma_1 + \gamma_2\Gamma_2 + \cdots + \gamma_m\Gamma_m\| \\ =& \sum_{i=1}^{m} p_i\|\Gamma_i\|.\end{aligned}$$

By using (22) it leads to

$$\begin{aligned}\sum_{k=1}^{\infty}\mathbb{E}\|\Delta U_k\| =& \sum_{k=1}^{\infty}(\mathbb{E}\|X_{k-1}\|)^k\,\mathbb{E}\|\Delta U_0\| \\ =& \sum_{k=1}^{\infty}\left(\sum_{i=1}^{m} p_i\|\Gamma_i\|\right)^k \mathbb{E}\|\Delta U_0\| \\ <& \sum_{k=1}^{\infty}\delta^k\mathbb{E}\|\Delta U_0\| \\ =& \frac{\delta}{1-\delta}\mathbb{E}\|\Delta U_0\| < \infty.\end{aligned}$$

Then by Markov inequality, for any $\epsilon > 0$ one has

$$\sum_{k=1}^{\infty} P(\|\Delta U_k\| > \epsilon) \le \sum_{k=1}^{\infty}\frac{\mathbb{E}\|\Delta U_k\|}{\epsilon} < \infty.$$

Therefore, one has $P(\|\Delta U_k\| > \epsilon, i.o.) = 0$ by Borel–Cantelli lemma, $\forall \epsilon > 0$, and then it leads to $P(\lim_{k\to\infty}\|\Delta U_k\| = 0) = 1$. That is, ΔU_k converges to zero almost surely. Noting that $\|E_k\| = \|\lfloor H\Delta U_k\rfloor_{N_k}\| \le \|H\Delta U_k\|$, the proof is completed. □

To show the mean square convergence, it is sufficient to show $\mathbb{E}\Delta U_k \Delta U_k^T \to 0$. That is, $F_k \to 0$. It is first noted that the matrix F_k recursively defined in (18) is positive definite. Then by the recurrence (18) one has the following theorem.

Theorem 3. *Consider system (2) with randomly varying iteration length and use control update law (8). The tracking error E_k converges to zero in mean square sense if the learning gain L satisfies (19).*

Proof. Following similar step of the proof of Theorem 2, there is a suitable constant $0 < \eta < 1$ such that

$$0 < \sum_{i=1}^{m} p_i\|\Gamma_i\|^2 < \eta.$$

Then one has

$$\begin{aligned}\|F_k\| =& \left\|\sum_{i=1}^{m} p_i\Gamma_i F_{k-1}\Gamma_i\right\| \\ \le& \sum_{i=1}^{m} p_i\|\Gamma_i F_{k-1}\Gamma_i\| \\ \le& \sum_{i=1}^{m} p_i\|\Gamma_i\|^2\|F_{k-1}\| \\ =& \left(\sum_{i=1}^{m} p_i\|\Gamma_i\|^2\right)\|F_{k-1}\| \\ <& \eta\|F_{k-1}\|.\end{aligned}$$

Thus the exponential zero-convergence of F_k is established and hence $F_k \to 0$, which means that $\Delta U_k \to 0$ in mean square sense. Similarly, by noting that $\|E_k\| \le \|H\Delta U_k\|$, the proof is completed. □

Remark 5. Generally speaking, convergence in almost sure sense and in mean square sense cannot be implied by each other. Thus it is hard to build convergence in these two senses meanwhile. The inherent reason that we could establish Theorems 2 and 3 for the same update law (8) is as follows. The convergence speed in mean square sense actually is exponential, as could be concluded from the proof of Theorem 3. Therefore, by direct calculations it is easy to find that $\sum_{k=0}^{\infty}\mathrm{Var}(\Delta U_k) < \infty$. Then, by Chebyshevs inequality and Borel–Cantelli lemma in probability theory, the almost sure convergence could be established.

5. Illustrative simulations

In order to show the effectiveness and robustness of the conventional P-type ILC algorithm, a time-varying system is given as follows

$$\begin{aligned}x(t+1,k) =& \begin{pmatrix}0.2\exp(-t/100) & -0.6 & 0\\ 0 & 0.50 & \sin(t)\\ 0 & 0 & 0.7\end{pmatrix}x(t,k) \\ &+ \begin{pmatrix}0\\ 0.3\sin(t)\\ 1.00\end{pmatrix}u(t,k)\\ y(t,k) =& (0 \quad 0.1 \quad 1.00 + 0.1\cos(t))x(t,k).\end{aligned}$$

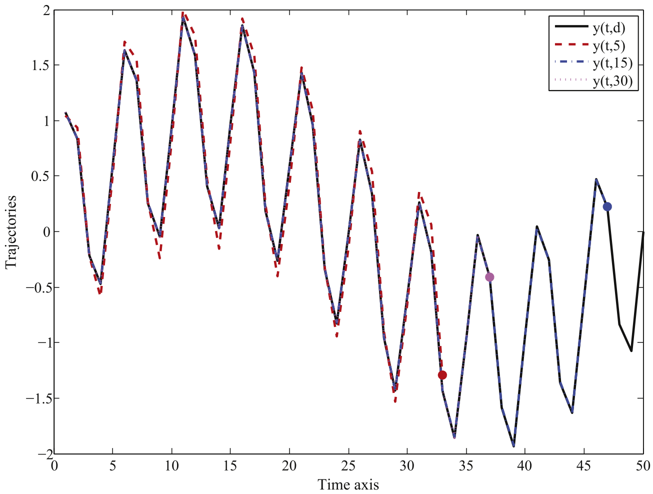

Fig. 1. The desired trajectory and tracking profiles for the 5-th, 15-th, and 30-th iterations.

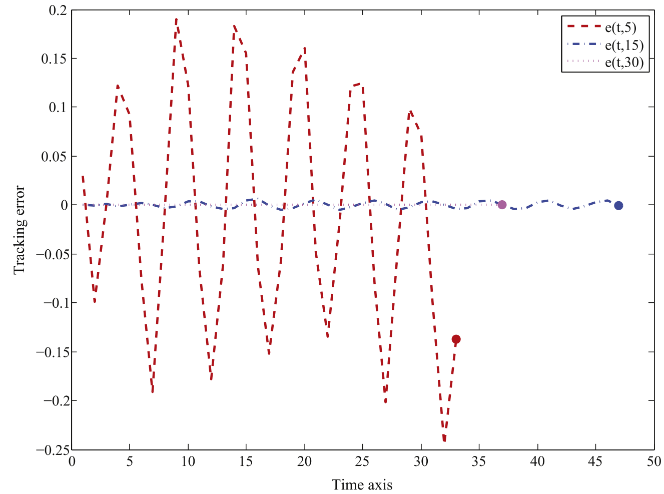

Fig. 2. Tracking error profiles for the 5-th, 15-th, and 30-th iterations.

The initial state is set as $x(0, k) = [0 \quad 0 \quad 0]^T$. Let the desired trajectory be $y(t, d) = \sin(2\pi t/50) + \sin(2\pi t/5)$. The maximum of iteration length is $N = 50$. Without loss of generality, the input of the initial iteration is simply set to zero, i.e., $u(t, 0) = 0,\ 0 \le t \le N$.

The iteration length varies from 30 to 50 satisfying discrete uniform distribution, which is just a simple case for illustration. The probability distribution is not required in our algorithm.

It is easy to find $C_{t+1}B_t = 1 + 0.03\sin(t) + 0.1\cos(t)$. The Arimoto-like gain is selected and $L_{t,t} = 0.5,\ \forall t$, therefore (19) is obviously valid. One could see from Fig. 1 that the output for the 5-th iteration has small deviations from the desired one, while for the 15-th iteration the output almost overlaps the desired trajectory. The convergence performance could also be further verified in Fig. 2, where the tracking error for these three iteration are displayed. One could see that the errors for the 15-th iteration are almost zero. It is noted that the lengths of the trajectories for the 5-th, 15-th, and 30-th iteration are less than 50. This demonstrates the fact that the iteration length varies iteration to iteration.

One may be interested in the convergence speed of the proposed P-type algorithm under randomly varying iteration lengths. This could be partially revealed from the solid blue line in Fig. 3, where the maximal error is defined by $\max_{1\le t\le N} \|e(t, k)\|$ for each iteration. As one could see, the maximal error decreases quickly as the iteration number increases. For a comparison, we also simulate the algorithm proposed in Li et al. (2014) and plot its maximal tracking error as the dashed red line. It is seen that the conventional P-type algorithm has a faster speed than the high-order ILC of Li et al. (2014). There are few explicit results on comparison of convergence speed between high-order ILC and the conventional P-type (Schmid, 2007). Here the reason that the P-type algorithm converges faster than the one in Li et al. (2014) might be that the conventional P-type has a quick response to large errors while the high-order scheme in Li et al. (2014) reduces the influence of large errors.

6. Conclusion

The tracking performance of traditional P-type ILC for linear system with randomly varying iteration lengths is discussed in this note. The probability properties along sample path are first calculated for further analysis. Then the zero-convergence both in almost sure sense and mean square sense is established, as long as the probability of full-length iteration is not zero. The sufficient condition on the design of learning gain is also clarified. For further study, the nonlinear system case is of great interest.

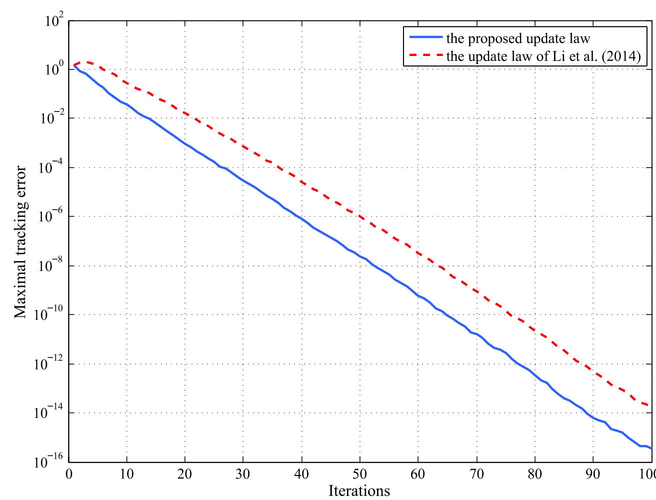

Fig. 3. Maximal tracking error $\max_{1\le t\le N} \|e(t, k)\|$ along iterations. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

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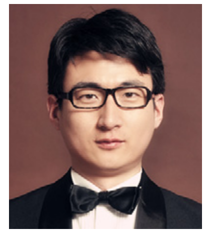

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