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Iterative Learning Control for discrete nonlinear systems with randomly iteration varying lengths[☆]

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ABSTRACT

This note proposes ILC for discrete-time affine nonlinear systems with randomly iteration varying lengths. No prior information on the probability distribution of random iteration length is required prior for controller design. The conventional P-type update law is used with a modified tracking error because of randomly iteration varying lengths. A novel technical lemma is proposed for the strict convergence analysis in pointwise sense. An illustrative example verifies the theoretical results.

1. Introduction

Iterative Learning Control (ILC) is a kind of intelligent control approach that is suitable for controlled systems completing given task in a finite interval repeatedly. The inherent idea of ILC is learning from past experiences and performing to the current process. To be specific, in ILC, the control signal is updated iteratively using information generated from previous iterations, so that the system output could track the desired trajectory asymptotically along the iteration index. The concept of ILC is first proposed by Arimoto in [1] driven from human learning ability to robotic systems. As has been developed for three decades, ILC has become a hot field of intelligent control theory, which is fruitful both in theoretical analysis and practical applications [2–4]. However, in most of the reported results, the operation length and reference trajectory are usually unchanged in different iterations, so that the update law could improve the tracking performance gradually. This condition may limit the applicability of ILC, and it motivates us to consider ILC problem under iteration varying factors.

Some previous publications have discussed the problem of ILC with varying references. Saab et al. studied ILC for continuous-time nonlinear systems with slowly varying references in [5], where D-type, PD-type, and PID-type update law were used to generate the control signal for tracking problem, respectively. The reference of each iteration was assumed to have a small deviation with that of the previous iteration. Xu proposed a direct learning control approach in [6,7] to handle two cases of varying references. One case is that the references have an identical spacial pattern but different time scales, and while the other one is that the references have an identical time scale but different magnitudes scales. Recently, in [8] Chi et al. proposed an adaptive ILC approach to cope with a class of high-order discrete-time system with the references being iteration-varying. The authors provided iteration-recursive algorithms to estimate parameters and generate corresponding input signals. However, the iteration length is still unchanged in these studies except [7].

Thus one is interested in the tracking ability when the operation length varies randomly during different iterations. This situation also exists in some ILC applications [9,10]. The biomedical systems, functional electrical stimulation for upper limb movement and for gait assistance, were given in [9], and a humanoid robot study was provided in [10]. In these equipments, the learning process could not be with the same length every iteration because of complex factors and unknown dynamics. As one could see, the iteration-varying length would lead to varying outputs, which further results in different signals in different iterations. To cope with this issue, [11] introduced an iteration-average operator and then designed an ILC algorithm for discrete linear systems. It was

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shown that the tracking error and input error would converge to zero in mathematical expectation sense. However, few results on nonlinear system are reported.

This note proceeds to consider the ILC problem for nonlinear discrete-time systems with iteration varying lengths. There are three major differences between [11] and the current paper. First of all, an iteration-average operator was introduced in [11] for update law design, and thus all historical data should be stored for sustained updating. In this paper, the conventional P-type update law is used only with a modified tracking error. We will show that the simple P-type ensures a good tracking performance. Moreover, the conventional λ-norm technique was used in [11], while in this paper, we make a modification on the definition of λ-norm so that it becomes more appropriate for the randomly varying length problem. Last but not least, a stronger convergence result is obtained in this note. In [11], the expectation of the tracking error is shown to converge to zero, while this note provides the almost sure convergence of the tracking error. To be specific, the zero-error tracking performance is proved when the initial state is accurately reset, while for the case of initial shifts, it is shown that the tracking error is bounded in proportion to the bound of initial state bias.

In practical applications, the actual operation length may be greater or smaller than the expected length. If the actual length is greater than the expected length, then the redundant signals are discarded as none information could be gotten from this signals. Therefore, this case could be regarded as the full-length case as long as one directly cut the trajectory at the position of expected length. If the actual length is smaller than the expected length, then the signals at the missing time instances cannot be obtained, thus no information could be used to update the input. As a result, for expression to be concise, only the case that the actual length is no greater than the expected one is taken into account in this note.

The rest of the note is arranged as follows: Section 2 gives the problem formulation; Section 3 presents the design of ILC algorithm, while its convergence analysis is provided in Section 4; illustrative simulations are shown in Section 5 and Section 6 concludes this note.

Notations: $\|M\|$ denotes the Euclidean norm of a square matrix M. $\sigma(M)$ is the eigenvalue of M. $\mathbb{R}$ is the set of real numbers, while $\mathbb{R}^m$ is the m-dimensional space. $\mathbb{E}(\cdot)$ and $\mathbb{P}(\cdot)$ denote the mathematical expectation and probability, respectively. $\|\theta(t)\|_\lambda$ denotes the λ-norm of a vector $\theta(t)$, $\lambda > 0$, which is defined as $\|\theta(t)\|_\lambda = \sup_{t\in\mathcal{S}} \alpha^{-\lambda t}\mathbb{E}\|\theta(t)\|$ where $\alpha > 1$ is a suitable selected constant and $\mathcal{S}$ is a finite discrete set of t. $I_{n\times n}$ denotes the unit matrix with dimension $n \times n$, and the subscript $n \times n$ may be omitted when no misunderstanding is caused. Let $\mathbf{1}(event)$ be an indicator function meaning that it equals 1 if the event indicated in the bracket is fulfilled, and 0 if the event does not hold.

2. Problem formulation

Consider the following discrete affine nonlinear system

$$\begin{aligned} x_k(t+1) &= f(x_k(t)) + Bu_k(t) \\ y_k(t) &= Cx_k(t) \end{aligned} \tag{1}$$

where $k = 0, 1, \ldots$ denotes iteration index, t is time instance, $t \in \{0, 1, \ldots, N_d\}$, and N_d is the expected iteration length. $x_k(t) \in \mathbb{R}^n$, $u_k(t) \in \mathbb{R}^p$, and $y_k(t) \in \mathbb{R}^q$ denote state, input, and output, respectively. f is the nonlinear function. C and B are matrices with appropriate dimensions. Without loss of generality, it is assumed that CB is of full-column-rank.

Remark 1. Matrices B and C are assumed time-invariant in system (1) to make the expressions concise. They can be extended to the time-varying case, $B(t)$ and $C(t)$, and/or state dependent case, $B(x(t))$, without making any further effort (see the analysis details below). Moreover, it will be shown that the convergence condition is independent of $f(\cdot)$ in the following. This is the major advantage of ILC, that is, ILC focuses on the convergence property along iteration axis and requires little system information. In addition, it is evident that the nonlinear function could be time varying.

Let $y_d(t)$, $t \in \{0, 1, \ldots, N_d\}$ be the desired trajectory. $y_d(t)$ is assumed to be realizable, that is, there is a suitable initial state $x_d(0)$ and unique input $u_d(t)$ such that

$$\begin{aligned} x_d(t+1) &= f(x_d(t)) + Bu_d(t) \\ y_d(t) &= Cx_d(t). \end{aligned} \tag{2}$$

The following assumptions are required for the technical analysis.

A1. The nonlinear function $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ satisfies global Lipschitz condition, that is, $\forall x_1, x_2 \in \mathbb{R}^n$,

$$\|f(x_1) - f(x_2)\| \le k_f \|x_1 - x_2\| \tag{3}$$

where $k_f > 0$ is the Lipschitz constant.

The global Lipschitz condition on nonlinear function is somewhat strong, although it is common in the ILC field for nonlinear systems. However, it should be pointed out that this assumption is imposed to facilitate the convergence derivations using λ-norm technique. With more efforts, the assumption could be extended to local Lipschitz case or continuous case [12,13].

A2. The identical initial condition is fulfilled, i.e., $x_k(0) = x_d(0)$, $\forall k$.

The initial state may not be reset precisely every iteration in practical applications, but the bias is usually bounded. Thus one would relax the assumption A2 to the following one.

A3. The initial state could shift from $x_d(0)$ but should be bounded, i.e., $\|x_d(0) - x_k(0)\| \le \epsilon$ where ϵ is a positive constant.

Let N_d denote the expected length. The actual length, N_k, varies in different iterations randomly. Thus, two cases need to be taken into account, i.e., $N_k < N_d$ and $N_k \ge N_d$. For the latter case, it is observed that only the data at the first N_d time instances are used for input updating. In a consequence, without loss of any generality, one could regard the latter case as $N_k = N_d$. From another point of view, one could regard N_d as the maximum length of actual lengths. For the former case, the outputs at the time instance $N_k + 1, \ldots, N_d$ are missing, and therefore, they are not available for updating. In other words, the input signals for the former N_k time instances are only updated.

The control objective of this note is to design ILC algorithm to track the desired trajectory y_d, $t \in \{0, 1, \ldots, N_d\}$, based on the available output $y_k(t)$, $t \in \{0, 1, \ldots, N_k\}$, $N_k \le N_d$, such that the tracking error $e_k(t)$, $\forall t$ converges to zero with probability one as the iteration number k goes to infinity.

The following lemma is needed for the following analysis, and its proof is put in the Appendix.

Lemma 1. *Let η be a Bernoulli binary random variable with $\mathbb{P}(\eta = 1) = \overline{\eta}$ and $\mathbb{P}(\eta = 0) = 1 - \overline{\eta}$. M is a positive matrix. Then the equality $\mathbb{E}\|I - \eta M\| = \|I - \overline{\eta}M\|$ holds if and only if one of the following conditions is satisfied:* (1) $\overline{\eta} = 0$; (2) $\overline{\eta} = 1$; *and* (3) $0 < \overline{\eta} < 1$ *and* $0 < M \le I$.

3. ILC design

In this note, the minimum length is denoted by N_m. Then the operation length varies among the discrete integer set $\{N_m, \ldots, N_d\}$, that is, the outputs at time $t = 0, 1, \dots, N_m$ are always available for input updating, while the availability of outputs at time $t = N_m + 1, \dots, N_d$ are random.

To describe the randomness of iteration length, we denote the probability of the occurrence of the output at time t by $p(t)$. Then it is found from the above explanations that $p(t) = 1, 0 \le t \le N_m$, and $0 < p(t) < 1, N_m + 1 \le t \le N_d$. Moreover, when the output at time t_0 is available in an iteration, the outputs at any time t where $t < t_0$ are definitely available in the same iteration. This further implies that $p(N_m) > p(N_m+1) > \cdots > p(N_d)$. It is worth pointing out that the probability is defined on time instance directly instead of on iteration length.

Note that the kth iteration length is denoted by N_k, which is a random variable valued in $\{N_m, \dots, N_d\}$. Let $\mathcal{A}_{N_k}$ be the event that the kth iteration length is N_k. Moreover, when an iteration length is N_k, it means the outputs at time $0 \le t \le N_k$ are available while the outputs at time $N_k + 1 \le t \le N_d$ are missing. Therefore, the probability of the kth iteration length being N_k is calculated as $\mathbb{P}(\mathcal{A}_{N_k}) = p(N_k) - p(N_{k+1})$. As a result, $\sum_{t=N_m}^{N_d} \mathbb{P}(\mathcal{A}_t) = 1$.

Remark 2. In [11], the probability of random iteration length is first given and then the probability of the output occurrence at each time instance is calculated. In this note, the calculation order is exchanged, that is, the probability of the output occurrence at each time instance is first given and then calculate the probability of random iteration length. However, the internally logical relationships are identical.

As long as $N_k < N_d$, the actual output information is not complete, i.e., the data of the former N_k time instances are only used to calculate tracking error for input updating. While for the left time instances, the input updating has to be suspended until the corresponding output information is available. In this case, we simply set the tracking error to be zero because none knowledge is obtained. In other words, a modified tracking error is defined as follows,

$$e_k^*(t) = \begin{cases} e_k(t), & 0 \le t \le N_k \\ 0, & N_k + 1 \le t \le N_d \end{cases} \tag{4}$$

where $e_k(t) \triangleq y_d(t) - y_k(t)$ is the original tracking error.

To make a more concise expression, let us introduce an indicator function $\mathbf{1}(t \le N_k)$. Then (4) could be reformulated as

$$e_k^*(t) = \mathbf{1}(t \le N_k) e_k(t). \tag{5}$$

Remark 3. For arbitrary given $t \le N_m$, the event $\{t \le N_k\}$ occurs with probability one. For arbitrary given $t > N_m$, the event $\{t \le N_k\}$ is a union of events $\{N_k = t\}, \{N_k = t + 1\}, \dots, \{N_k = N_d\}$. Thus the probability of the event $\{\mathbf{1}(t \le N_k) = 1\}$ is calculated as $\mathbb{P}(\mathbf{1}(t \le N_k) = 1) = \sum_{i=t}^{N_d} \mathbb{P}(\mathcal{A}_i) = p(t), t > N_m$. Combining these two scenarios, we have $\mathbb{P}(\mathbf{1}(t \le N_k) = 1) = p(t), \forall t$. In addition, $\mathbb{E}(\mathbf{1}(t \le N_k)) = \mathbb{P}(\mathbf{1}(t \le N_k) = 1) \times 1 + \mathbb{P}(\mathbf{1}(t \le N_k) = 0) \times 0 = p(t)$.

With the help of the modified tracking error, we can give the following update law for input signal now

$$u_{k+1}(t) = u_k(t) + L e_k^*(t + 1) \tag{6}$$

where L is the learning gain to be defined later, $L \in \mathbb{R}^{p \times q}$.

Remark 4. Generally speaking, to ensure a good performance against high-frequency signals in practical applications such as unmodeled dynamics, a low-pass Q-filter is incorporated in the leaning algorithm. In other words, (6) is formulated as $u_{k+1}(t) = Q(q)(u_k(t) + L e_k^*(t + 1))$, where $Q(q)$ denotes the Q-filter [2]. The involved Q-filter could suppress the high-frequency components and pass the low-frequency components. Therefore, it has effects on the convergence condition as shown in [2]. To make the analysis derivations concise, we only consider the case $Q(q) = I$ in the following of this note.

4. Convergence analysis

The following theorem gives the zero-error convergence of the proposed ILC algorithm for the case that initial state is accurately reset.

Theorem 1. *Consider discrete-time affine nonlinear system (1) and ILC algorithm (6), and assume A1 and A2 hold. If the learning gain matrix L satisfies that $0 < LCB < I$, then the tracking error would converge to zero as iteration number k goes to infinity, i.e., $\lim_{k\to\infty} e_k(t) = 0$, $t = 1, \dots, N_d$.*

Proof. Subtracting both sides of (6) from $u_d(t)$, one has

$$\delta u_{k+1}(t) = \delta u_k(t) - L e_k^*(t + 1) \tag{7}$$

where $\delta u_k(t) \triangleq u_d(t) - u_k(t)$ is the input error.

Noticing (1) and (2), it follows that

$$\begin{aligned} \delta x_k(t + 1) &= (f(x_d(t)) - f(x_k(t))) + B\delta u_k(t) \\ e_k(t) &= C\delta x_k(t) \end{aligned} \tag{8}$$

where $\delta x_k(t) \triangleq x_d(t) - x_k(t)$, which further leads to

$$\begin{aligned} e_k(t + 1) &= C\delta x_k(t + 1) \\ &= CB\delta u_k(t) + C(f(x_d(t)) - f(x_k(t))). \end{aligned} \tag{9}$$

Substitute (9) and (5) into (7),

$$\begin{aligned} \delta u_{k+1}(t) &= \delta u_k(t) - \mathbf{1}(t \le N_k) L e_k(t + 1) \\ &= \delta u_k(t) - \mathbf{1}(t \le N_k) L[CB\delta u_k(t) \\ &\quad + C(f(x_d(t)) - f(x_k(t)))] \\ &= (I - \mathbf{1}(t \le N_k) LCB)\delta u_k(t) \\ &\quad - \mathbf{1}(t \le N_k) LC(f(x_d(t)) - f(x_k(t))). \end{aligned}$$

Taking Euclidean norm of both sides of last equation, one has

$$\begin{aligned} \|\delta u_{k+1}(t)\| &\le \|(I - \mathbf{1}(t \le N_k) LCB)\delta u_k(t)\| \\ &\quad + \|\mathbf{1}(t \le N_k) LC(f(x_d(t)) - f(x_k(t)))\| \\ &\le \|(I - \mathbf{1}(t \le N_k) LCB)\| \, \|\delta u_k(t)\| \\ &\quad + \|\mathbf{1}(t \le N_k) LC\| \, \|(f(x_d(t)) - f(x_k(t)))\| \\ &\le \|(I - \mathbf{1}(t \le N_k) LCB)\| \, \|\delta u_k(t)\| \\ &\quad + k_f \|\mathbf{1}(t \le N_k) LC\| \, \|\delta x_k(t)\|. \end{aligned}$$

Noticing that the event $t \le N_k$ is independent of $\delta u_k(t)$ and $\delta x_k(t)$. Thus by taking mathematical expectation last inequality, it follows

$$\begin{aligned} \mathbb{E}\|\delta u_{k+1}(t)\| &\le \mathbb{E}(\|(I - \mathbf{1}(t \le N_k) LCB)\| \, \|\delta u_k(t)\|) \\ &\quad + k_f \mathbb{E}(\|\mathbf{1}(t \le N_k) LC\| \, \|\delta x_k(t)\|) \\ &\le \|(I - p(t) LCB)\| \mathbb{E}\|\delta u_k(t)\| \\ &\quad + k_f \|p(t) LC\| \mathbb{E}\|\delta x_k(t)\| \end{aligned} \tag{10}$$

where for the last inequality, Lemma 1 is used by noticing that $0 < LCB < I$.

On the other hand, take Euclidean norm of both sides of the first equation in (8),

$$\begin{aligned} \|\delta x_k(t + 1)\| &\le \|B\| \, \|\delta u_k(t)\| + \|f(x_d(t)) - f(x_k(t))\| \\ &\le \|B\| \, \|\delta u_k(t)\| + k_f \|x_d(t) - x_k(t)\| \\ &= \|B\| \, \|\delta u_k(t)\| + k_f \|\delta x_k(t)\| \end{aligned} \tag{11}$$

and then take mathematical expectation,

$$\mathbb{E}\|\delta x_k(t + 1)\| \le k_b \mathbb{E}\|\delta u_k(t)\| + k_f \mathbb{E}\|\delta x_k(t)\| \tag{12}$$

where $k_b \ge \|B\|$. Based on the recursion of (12) and noting A2, one has

$$\begin{aligned} \mathbb{E}\|\delta x_k(t + 1)\| &\le k_b \mathbb{E}\|\delta u_k(t)\| + k_f k_b \mathbb{E}\|\delta u_k(t - 1)\| \\ &\quad + k_f^2 \mathbb{E}\|\delta x_k(t - 1)\| \end{aligned}$$

$$\begin{aligned}&\le k_b\mathbb{E}\|\delta u_k(t)\|+k_bk_f\mathbb{E}\|\delta u_k(t-1)\|+\cdots\\&\quad+k_bk_f^{t-1}\mathbb{E}\|\delta u_k(1)\|+k_bk_f^t\mathbb{E}\|\delta u_k(0)\|\\&\quad+k_f^t\mathbb{E}\|\delta x_k(0)\|\\&=k_b\sum_{i=0}^{t}k_f^{t-i}\mathbb{E}\|\delta u_k(i)\| \qquad (13)\end{aligned}$$

which further infers

$$\mathbb{E}\|\delta x_k(t)\|\le k_b\sum_{i=0}^{t-1}k_f^{t-1-i}\mathbb{E}\|\delta u_k(i)\|. \qquad (14)$$

Then substituting (14) into (10) yields that

$$\begin{aligned}\mathbb{E}\|\delta u_{k+1}(t)\|&\le\|(I-p(t)LCB)\|\mathbb{E}\|\delta u_k(t)\|\\&\quad+k_b\|p(t)LC\|\sum_{i=0}^{t-1}k_f^{t-i}\mathbb{E}\|\delta u_k(i)\|. \qquad (15)\end{aligned}$$

Apply the λ-norm to both sides of last inequality, i.e., multiply both sides of last inequality with $\alpha^{-\lambda t}$ and take supremum according to all time instances t, then one has

$$\begin{aligned}&\sup_t\alpha^{-\lambda t}\mathbb{E}\|\delta u_{k+1}(t)\|\\&\quad\le\sup_t\|(I-p(t)LCB)\|\cdot\sup_t\alpha^{-\lambda t}\mathbb{E}\|\delta u_k(t)\|\\&\qquad+k_b\cdot\sup_t\|p(t)LC\|\cdot\sup_t\alpha^{-\lambda t}\left(\sum_{i=0}^{t-1}k_f^{t-i}\mathbb{E}\|\delta u_k(i)\|\right). \qquad (16)\end{aligned}$$

Let $\alpha > k_f$, then it is observed that

$$\begin{aligned}&\sup_t\alpha^{-\lambda t}\left(\sum_{i=0}^{t-1}k_f^{t-i}\mathbb{E}\|\delta u_k(i)\|\right)\\&\quad\le\sup_t\alpha^{-\lambda t}\left(\sum_{i=0}^{t-1}\alpha^{t-i}\mathbb{E}\|\delta u_k(i)\|\right)\\&\quad\le\sup_t\left(\sum_{i=0}^{t-1}\alpha^{-(\lambda-1)t-i}\mathbb{E}\|\delta u_k(i)\|\right)\\&\quad\le\sup_t\left(\sum_{i=0}^{t-1}\alpha^{-\lambda i}\mathbb{E}\|\delta u_k(i)\|\cdot\alpha^{-(\lambda-1)(t-i)}\right)\\&\quad\le\sup_t\left(\sum_{i=0}^{t-1}\sup_t\left(\alpha^{-\lambda i}\mathbb{E}\|\delta u_k(i)\|\right)\alpha^{-(\lambda-1)(t-i)}\right)\\&\quad\le\sup_t\left(\alpha^{-\lambda i}\mathbb{E}\|\delta u_k(i)\|\right)\sup_t\left(\sum_{i=0}^{t-1}\alpha^{-(\lambda-1)(t-i)}\right)\\&\quad\le\sup_t\left(\alpha^{-\lambda i}\mathbb{E}\|\delta u_k(i)\|\right)\times\frac{1-\alpha^{-(\lambda-1)N_d}}{\alpha^{\lambda-1}-1}.\end{aligned}$$

Therefore, from (16),

$$\begin{aligned}\|\delta u_{k+1}(t)\|_\lambda&\le\sup_t\|(I-p(t)LCB)\|\,\|\delta u_k(t)\|_\lambda\\&\quad+k_b\cdot\sup_t\|p(t)LC\|\,\|\delta u_k(t)\|_\lambda\times\frac{1-\alpha^{-(\lambda-1)N_d}}{\alpha^{\lambda-1}-1}\\&\le\left(\rho+k_b\varphi\frac{1-\alpha^{-(\lambda-1)N_d}}{\alpha^{\lambda-1}-1}\right)\|\delta u_k(t)\|_\lambda\end{aligned}$$

where ρ and φ are defined as

$$\rho=\sup_t\|(I-p(t)LCB)\|$$

$$\varphi=\sup_t\|p(t)LC\|.$$

Noticing that the learning gain matrix L satisfies $0 < LCB < I$ and $0 < p(t) \le 1, \forall t$, it is evident that $\|I-p(t)LCB\| < 1, \forall t$. Since $0 \le t \le N_d$ has only finite values, one has $0 < \rho < 1$. Let $\alpha > \max\{1, k_f\}$, then there always exists a λ large enough such that $k_b\varphi\frac{1-\alpha^{-(\lambda-1)N_d}}{\alpha^{\lambda-1}-1} < 1-\rho$, which further yields

$$\overline{\rho}\triangleq\rho+k_b\varphi\frac{1-\alpha^{-(\lambda-1)N_d}}{\alpha^{\lambda-1}-1}<1. \qquad (17)$$

This means

$$\lim_{k\to\infty}\|\delta u_k(t)\|_\lambda=0,\quad\forall t.$$

Again, by the finiteness of t, one has

$$\lim_{k\to\infty}\mathbb{E}\|\delta u_k(t)\|=0,\quad\forall t. \qquad (18)$$

Notice that $\|\delta u_k(t)\|\ge 0$, thus it could be concluded from (18) that

$$\lim_{k\to\infty}\|\delta u_k(t)\|=0,\quad\forall t. \qquad (19)$$

Then directly by mathematical induction method along t, it is easy to show that $\lim_{k\to\infty}\delta x_k(t)=0$ and $\lim_{k\to\infty}e_k(t)=0, \forall t$. The proof is thus completed. □

Remark 5. One may argue whether it is conservative to design L such that $0 < LCB < I$. In our point of view, it is a tradeoff between algorithm design and scope of application. In [11], the condition on L is somewhat loose; however, the occurrence probability of randomly varying length is required to be estimated prior because the convergence condition depends on it. While in this note, the requirements on L are a little restrictive, but no information on probability is requested prior. Thus it is more suitable for practical applications. Here, two simple schemes are referential if knowledge of CB is available. The first is that design L_* such that $L_*CB > 0$ and then multiply a constant μ small enough such that $\mu L_*CB < I$, whence $L=\mu L_*$. The second is that let $L=\frac{(CB)^T}{\beta+\|CB\|^2}$, where $\beta > 0$.

Remark 6. The operation length varies from iteration to iteration, thus one may doubt why Theorem 1 claims that the tracking error would converge to zero for all time instances. In other words, the influence of random varying length is not revealed. We have some explanations for this issue. On one hand, in the proof, we introduce the so-called λ-norm, which is similar to the conventional λ-norm in earlier publications but is modified with an additional expectation to deal with the randomness, eliminate the random indicator function $\mathbf{1}(t\le N_k)$ and convert the original expression into deterministic case. On the other hand, there is a positive probability that the iteration length achieves the maximal length N_d, thus the input at each time instance would be updated more or less. To be specific, the input at time instance $t \le N_m$ would be updated for all iterations, while the input at time instance $N_m < t \le N_d$ is updated for part iterations. Therefore, the input at different time instances may have different convergence speed but they all converge to the desired one.

Remark 7. The time-invariant model (1) is studied in this note. However, the system is easy to extend to time-varying case, where the design condition is slightly modified as $0 < L(t)C(t+1)B(t) < I$. The convergence analysis could be completely same to the one in this note. Moreover, the P-type update law (6) could also be extended to other types of ILC such as PD-type ILC with slight modifications to the proof and convergence conditions. Furthermore, the considered system is of relative degree 1, that is, CB is of full column rank. This leads us to design the P-type update law (6) and convergence condition $0 < LCB < I$ in the theorem. Under some circumstances, the system may be of high relative degree τ, that is, $C\frac{\partial f^{\tau-1}(f(x)+Bu)}{\partial u}$ is of full column rank and $C\frac{\partial f^{i}(f(x)+Bu)}{\partial u}=0$, $0\le i\le \tau-2$, where $f^i(x)=f^{i-1}\circ f(x)$ and $\circ$ denotes the composite operator of functions [14]. For this case, the analysis is still valid provided that the update law is modified as $u_{k+1}(t)=u_k(t)+Le_k^*(t+\tau)$ and the convergence condition becomes $0<LC\frac{\partial f^{\tau-1}(f(x)+Bu)}{\partial u}<I$.

Remark 8. We provide a convergence analysis in modified λ-norm sense above for the ILC problem for discrete nonlinear systems under randomly iteration-varying length situation. One may interest in monotonic convergence in vector norm sense. To this end, define the lifted sup-vector $U_k\triangleq[\mathbb{E}\|\delta u_k(0)\|^T,\mathbb{E}\|\delta u_k(1)\|^T,\ldots,\mathbb{E}\|\delta u_k(N-1)\|^T]^T$ and the associated matrix Γ from (15) as a block lower-triangular matrix with its elements being the parameters of (15), then we have $\|U_{k+1}\|_1\le\|\Gamma\|_\infty\|U_k\|_1$ from (15) directly, where $\|\cdot\|_1$ and $\|\cdot\|_\infty$ denote the 1-norm of a vector and the ∞-norm of a matrix, respectively. Consequently, we draw a conclusion that the input error converges to zero monotonically if one can design L such that $\|\Gamma\|_\infty<1$.

The identical initial condition A2 is required to make the analysis more concise. However, it is of great interest to consider the case that the i.i.c. is no longer valid. To be specific, the initial state may vary in a small zone, then it could be proven that the tracking error would converge into a small zone, of which the bound is in proportion to initial state error. This is given in the next theorem.

Theorem 2. *Consider discrete-time affine nonlinear system (1) and ILC algorithm (6), and assume A1 and A3 hold. If the learning gain matrix L satisfies that $0<LCB<I$, then the tracking error would converge to small zone, whose bound is in proportion to ϵ, as iteration number k goes to infinity, i.e., $\limsup_{k\to\infty}\mathbb{E}\|e_k(t)\|\le\gamma\epsilon$, $t=1,\ldots,N_d$, where γ is a suitable constant.*

Proof. The proof follows the one of Theorem 1 with minor technical modifications. The derivations from (7) to (12) remain unchanged. While (13) is replaced by

$$\mathbb{E}\|\delta x_k(t+1)\|\le k_b\sum_{i=0}^{t}k_f^{t-i}\mathbb{E}\|\delta u_k(i)\|+k_f^t\mathbb{E}\|\delta x_k(0)\|. \tag{20}$$

Combining with A3 it leads to

$$\mathbb{E}\|\delta x_k(t)\|\le k_b\sum_{i=0}^{t-1}k_f^{t-1-i}\mathbb{E}\|\delta u_k(i)\|+k_f^{t-1}\epsilon. \tag{21}$$

Then substituting (21) into (10) yields that

$$\begin{aligned}\mathbb{E}\|\delta u_{k+1}(t)\|\le{}&\|(I-p(t)LCB)\|\mathbb{E}\|\delta u_k(t)\|\\&+k_b\|p(t)LC\|\sum_{i=0}^{t-1}k_f^{t-i}\mathbb{E}\|\delta u_k(i)\|\\&+\|p(t)LC\|k_f^t\epsilon.\end{aligned} \tag{22}$$

Apply the λ-norm to both sides of last inequality, and by similar steps to the proof of Theorem 1, one is easy to get

$$\|\delta u_{k+1}(t)\|_\lambda\le\overline{\rho}\|\delta u_k(t)\|_\lambda+\sup_t\alpha^{-\lambda t}\|p(t)LC\|k_f^t\epsilon. \tag{23}$$

By the finiteness of t, there is a constant ϑ such that $\sup_t\alpha^{-\lambda t}\|p(t)LC\|k_f^t<\vartheta$ and then

$$\|\delta u_{k+1}(t)\|_\lambda\le\overline{\rho}\|\delta u_k(t)\|_\lambda+\vartheta\epsilon \tag{24}$$

which further means

$$\limsup_{k\to\infty}\|\delta u_{k+1}(t)\|_\lambda\le\frac{\vartheta\epsilon}{1-\overline{\rho}}. \tag{25}$$

This implies that

$$\limsup_{k\to\infty}\mathbb{E}\|\delta u_{k+1}(t)\|\le\frac{\alpha^{\lambda t}\vartheta\epsilon}{1-\overline{\rho}}.$$

Then combining with (21) one is easy to find that $\mathbb{E}\|\delta x_k(t)\|$ is bounded in proportion to ϵ, and thus a suitable γ exists such that $\limsup_{k\to\infty}\mathbb{E}\|e_k\|\le\gamma\epsilon$. This completes the proof. □

5. Illustrative simulations

In order to show the effectiveness of the proposed ILC algorithm and verify the convergence analysis, consider the following affine nonlinear system

$$\begin{aligned}x_k^{(1)}(t+1)&=\cos(x_k^{(1)}(t))+0.3x_k^{(2)}(t)x_k^{(1)}(t)\\x_k^{(2)}(t+1)&=0.4\sin(x_k^{(1)}(t))+\cos(x_k^{(2)}(t))+u_k(t)\\y_k(t)&=x_k^{(2)}(t)\end{aligned}$$

which means $B=[0\ 1]^T$ and $C=[0\ 1]$ in (1). $x_k(t)=[x_k^{(1)}(t),x_k^{(2)}(t)]^T$ denotes the state.

The expected iteration length is $N_d=50$. To simulate the randomly iteration-varying length, let $N_m=40$. In other words, the iteration length N_k varies from 40 to 50. As a simple case for illustration, we let N_k satisfy discrete uniform distribution during the discrete set $\{40,41,\ldots,50\}$. It should be noted that the probability distribution is not required for the control design. However, the probability distribution would alter the convergence speed. This is because, generally speaking, larger probability means more updates to the corresponding input, which therefore leads to faster convergence. If $\mathbb{P}(N_d)$ is very close to 1, that is, most iterations could complete the maximum length, then the behavior along iteration axis would be very close to the iteration-length-invariant tracking case and thus a fast convergence speed could be obtained.

The desired tracking trajectory is

$$y_d(t)=0.8\sin\left(\frac{2\pi t}{50}\right)+2\sin\left(\frac{2\pi t}{25}\right)+\sin\left(\frac{\pi t}{5}\right).$$

The initial state is set to be $x_k(0)=[0,0]^T$. Without loss of generality, the input of the initial iteration is zero, i.e., $u_0(t)=0$, $0\le t\le N_d$. It is obvious that CB is equal to 1, thus we set the learning gain in (6) as 0.5. The algorithm runs for 50 iterations. The desired trajectory and output at 50th iteration are shown in Fig. 1, where the red solid line denotes the desired trajectory, and the blue dashed line marked with circles denotes the output at the 50th iteration. As one could see, the system output achieves perfect tracking performance.

The tracking error profiles of the whole time interval at the 15th, 20th, 30th, and 40th iterations are shown in Fig. 2. It is observed that the error at the 15th iteration has been small. At the 20th iteration, the tracking errors are already acceptable. Meanwhile, the error profiles of different iterations end at different time instants, which demonstrates the random varying iteration length circumstance.

The convergence property along iterations is shown in Fig. 3, illustrated by the blue line marked with circles, where the maximal tracking error is defined by $\max_t|e_k(t)|$ for the kth iteration. As commented in Remark 7, the proposed algorithm could be extended to PD-type algorithm. Here we also make simulations based on PD-type update law $u_{k+1}(t)=u_k(t)+L_pe_k^*(t+1)+K_d(e_k^*(t+1)-e_k^*(t))$ with learning gain $L_p=0.4$ and $K_d=0.3$. The maximal tracking error profile along iteration axis is illustrated by the red line marked with triangles. As one could see from Fig. 3, the maximal tracking error reduces to zero fast.

To verify the convergence under varying initial states, we let each dimension of the initial state obeys a uniform distribution

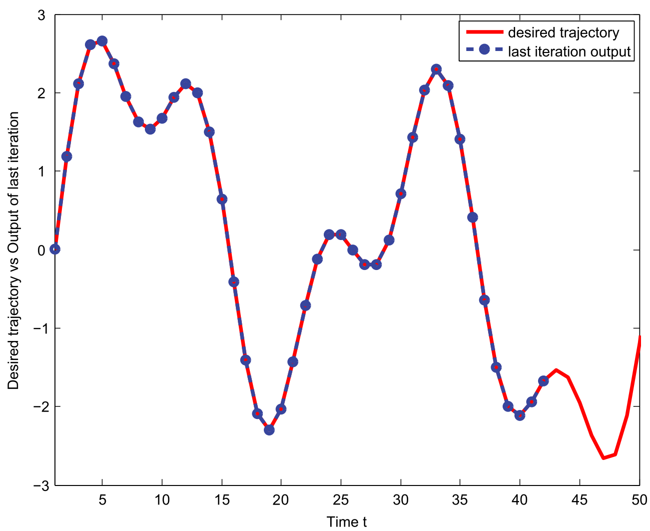

Fig. 1. Tracking Performance of the output at the 50th iteration. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

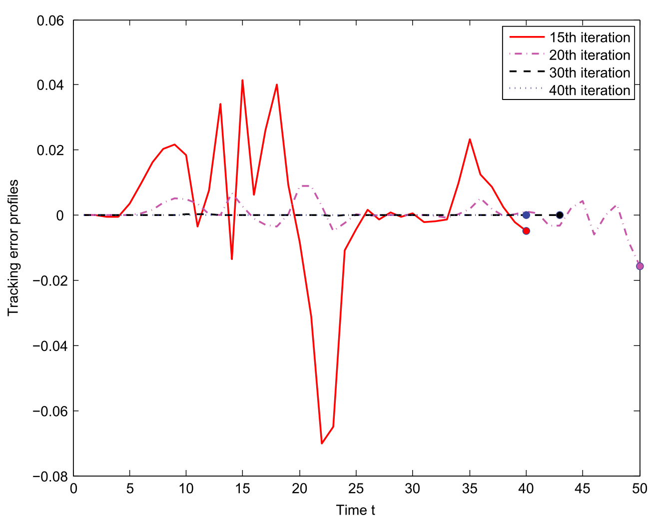

Fig. 2. Tracking error profiles at the 15th, 20th, 30th, and 40th iterations.

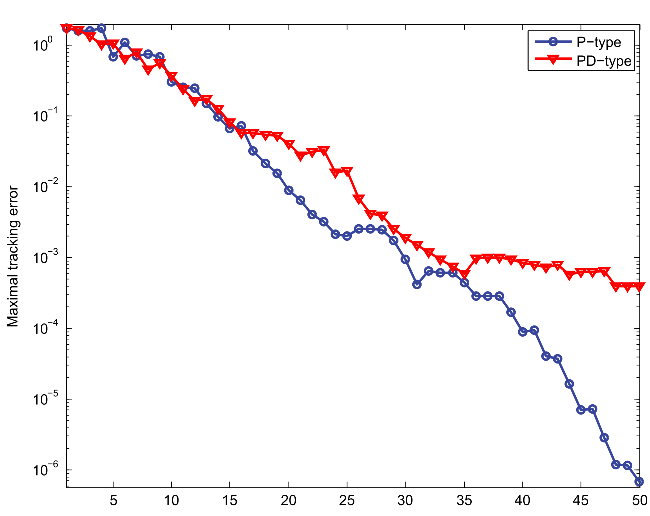

Fig. 3. Maximal errors along iterations. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

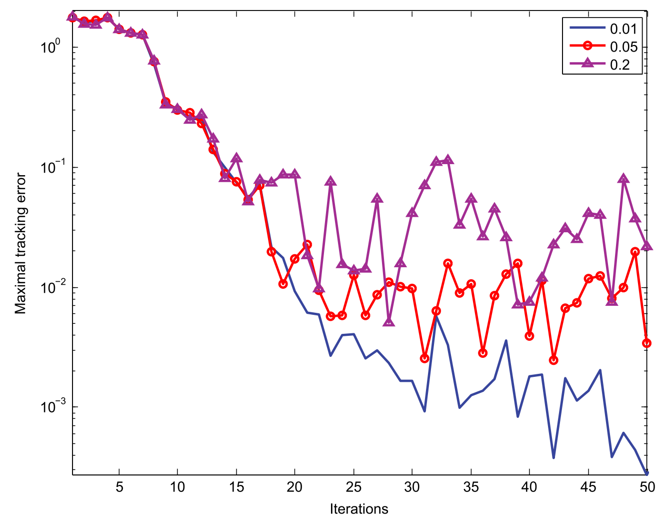

Fig. 4. Maximal errors for the case with iteration-varying initial value.

in $[-\epsilon, \epsilon]$ with different scales ϵ being 0.01, 0.05, and 0.2. The tracking performance would be inferior to the identical initial condition case. However, the algorithm still maintains a robust performance, as shown in Fig. 4, where one could find that large initial bias leads to large bound of tracking errors.

We conclude this section with several remarks. The simulations have shown that the conventional P-type update law has a good performance against randomly iteration varying lengths for discrete-time affine nonlinear systems. Although the convergence in λ-norm sense does not imply the monotonic decreasing naturally, the simulations show that the tracking performance is sustainedly improved. Moreover, when encountering other practical issues, the proposed P-type algorithm can be modified by incorporating other design techniques.

6. Conclusion

This note proposes the first convergence analysis of ILC for discrete-time affine nonlinear systems with randomly iteration varying lengths. A random variable is introduced to describe the random length. Then the tracking error is modified to facilitate the practical situations. The traditional P-type update law is taken as the control algorithm for our research and it can be extended to other schemes. If the identical initial condition is satisfied, the tracking error is proved to converge to zero as the iteration number goes to infinity by using modified supremum norm technique. If the initial state shifts in a small bound, then it is shown that the tracking error is also bounded. It is worth pointing out that the probability of random length is not required prior for control design. Due to the usage of modified supremum norm technique, the nonlinear function in this paper is required to satisfy globally Lipschitz condition. The extension to locally Lipschitz condition case can be made by introducing different analysis technique and will be reported in the future. For further study, the case of general nonlinear systems, especially those allow nonlinearities in the actuators and/or sensors, are of great interest. It has been shown that the conventional P-type update law performs well for these nonlinear systems, thus it is of great possibility that the scheme proposed in this note could solve the tracking problem for general nonlinear systems under randomly iteration varying lengths.

Appendix

Proof of Lemma 1. Let us first prove the sufficiency. It is easy to see that if $\overline{\eta} = 0$ or $\overline{\eta} = 1$, which means $\eta \equiv 0$ and $\eta \equiv 1$, respectively, then the equation $\mathbb{E}\|I - \eta M\| = \|I - \overline{\eta} M\|$ is valid. Moreover, the equality also holds obviously if $M = I$. Thus it is sufficient to show the equation for the case $0 < \overline{\eta} < 1$ and $0 < M < I$. By the definition of mathematical expectation for discrete random variables, one has

$$\begin{aligned}\mathbb{E}\|I - \eta M\| &= \|I - 0 \cdot M\| \cdot \mathbb{P}(\eta = 0) + \|I - 1 \cdot M\| \cdot \mathbb{P}(\eta = 1)\\ &= (1 - \overline{\eta}) + \overline{\eta}\|I - M\|\\ &= 1 + \overline{\eta}(\|I - M\| - 1).\end{aligned}$$

Noticing that M is a positive matrix and $0 < M < I$, $I - M$ is a positive matrix. Moreover, for a positive matrix, the Euclidean norm is equal to its maximal eigenvalue, i.e., $\|I - M\| = \sigma_{\max}(I - M)$, and therefore, $\|I - M\| = 1 - \sigma_{\min}(M)$. This further leads to

$$\mathbb{E}\|I - \eta M\| = 1 - \overline{\eta}\sigma_{\min}(M).$$

On the other hand, noting $0 < \overline{\eta} < 1$,

$$\begin{aligned}\|I - \overline{\eta} M\| &= \sigma_{\max}(I - \overline{\eta} M)\\ &= 1 - \sigma_{\min}(\overline{\eta} M)\\ &= 1 - \overline{\eta}\sigma_{\min}(M).\end{aligned}$$

Next, for the necessity, it only needs to show that the equality $\mathbb{E}\|I - \eta M\| = \|I - \overline{\eta} M\|$ is not valid if $M > I$ and $0 < \overline{\eta} < 1$. In this case, it is easy to find

$$\begin{aligned}\mathbb{E}\|I - \eta M\| &= 1 + \overline{\eta}(\|I - M\| - 1)\\ &= 1 + \overline{\eta}(\sigma_{\max}(M - I) - 1)\\ &= 1 + \overline{\eta}\sigma_{\max}(M) - 2\overline{\eta}\end{aligned}$$

while the norm $\|I - \overline{\eta} M\|$ is complex as three cases should be discussed respectively.

(a) If $I - \overline{\eta} M$ is negative definite, i.e., $I - \overline{\eta} M < 0$, then $\|I - \overline{\eta} M\| = \overline{\eta}\sigma_{\max}(M) - 1$;
(b) If $I - \overline{\eta} M$ is positive definite, i.e., $I - \overline{\eta} M > 0$, then $\|I - \overline{\eta} M\| = 1 - \overline{\eta}\sigma_{\min}(M)$;
(c) If $I - \overline{\eta} M$ is indefinite, then $\|I - \overline{\eta} M\| = \max\{\overline{\eta}\sigma_{\max}(M) - 1, 1 - \overline{\eta}\sigma_{\min}(M)\}$.

Thus it is sufficient to verify that $\mathbb{E}\|I - \eta M\|$ equal neither $\overline{\eta}\sigma_{\max}(M) - 1$ nor $1 - \overline{\eta}\sigma_{\min}(M)$. Suppose $\mathbb{E}\|I - \eta M\| = \overline{\eta}\sigma_{\max}(M) - 1$, then one has $1 + \overline{\eta}\sigma_{\max}(M) - 2\overline{\eta} = \overline{\eta}\sigma_{\max}(M) - 1$, which means $\overline{\eta} = 1$, and this contradicts with $0 < \overline{\eta} < 1$. Otherwise, suppose $\mathbb{E}\|I - \eta M\| = 1 - \overline{\eta}\sigma_{\min}(M)$, then one has $1 + \overline{\eta}\sigma_{\max}(M) - 2\overline{\eta} = 1 - \overline{\eta}\sigma_{\min}(M)$, which means $\sigma_{\max}(M) + \sigma_{\min}(M) = 2$, and this contradicts with $M > I$.

The proof is thus completed. □

References

[1] S. Arimoto, S. Kawamura, F. Miyazaki, Bettering operation of robots by learning, J. Robot. Syst. 1 (2) (1984) 123–140.
[2] D.A. Bristow, M. Tharayil, A.G. Alleyne, A survey of iterative learning control: A learning-based method for high-performance tracking control, IEEE Control Syst. Mag. 26 (3) (2006) 96–114.
[3] H.S. Ahn, Y.Q. Chen, K.L. Moore, Iterative learning control: survey and categorization from 1998 to 2004, IEEE Trans. Syst. Man. Cybern. C 37 (6) (2007) 1099–1121.
[4] D. Shen, Y. Wang, Survey on stochastic iterative learning control, J. Process Control 24 (12) (2014) 64–77.
[5] S.S. Saab, W.G. Vogt, M.H. Mickle, Learning control algorithms for tracking slowly varying trajectories, IEEE Trans. Syst. Man Cybern. Part B: Cybern. 27 (4) (1997) 657–679.
[6] J.-X. Xu, Direct learning of control efforts for trajectories with different magnitudes scales, Automatica 33 (12) (1997) 2191–2195.
[7] J.-X. Xu, Direct learning of control efforts for trajectories with different time scales, IEEE Trans. Automat. Control 43 (7) (1998) 1027–1030.
[8] R. Chi, Z. Hou, J.-X. Xu, Adaptive ILC for a class of discrete-time systems with iteration-varying trajectory and random initial condition, Automatica 44 (8) (2008) 2207–2213.
[9] T. Seel, T. Schauer, J. Raisch, Iterative learning control for variable pass length systems, in: Proceedings of the 18th IFAC World Congress, August 28–September 2, Milano, Italy, 2011, pp. 4880–4885.
[10] R.W. Longman, K.D. Mombaur, Investigating the use of iterative learning control and repetitive control to implement periodic gaits, Lecture Notes in Control and Inform. Sci. 340 (2006) 189–218.
[11] X. Li, J.-X. Xu, D. Huang, An iterative learning control approach for linear systems with randomly varying trial lengths, IEEE Trans. Automat. Control 59 (7) (2014) 1954–1960.
[12] D. Shen, Y. Wang, Iterative learning control for networked stochastic systems with random packet losses, Internat. J. Control 88 (5) (2015) 959–968.
[13] D. Shen, Y. Wang, ILC for networked nonlinear systems with unknown control direction through random Lossy channel, Systems Control Lett. 77 (2015) 30–39.
[14] M.X. Sun, D. Wang, Analysis of nonlinear discrete-time systems with higher-order iterative learning control, Dynam. Control 11 (2001) 81–96.