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RESEARCH ARTICLE

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A framework of iterative learning control under random data dropouts: Mean square and almost sure convergence

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Summary

This paper addresses the iterative learning control problem under random data dropout environments. The recent progress on iterative learning control in the presence of data dropouts is first reviewed from 3 aspects, namely, data dropout model, data dropout position, and convergence meaning. A general framework is then proposed for the convergence analysis of all 3 kinds of data dropout models, namely, the stochastic sequence model, the Bernoulli variable model, and the Markov chain model. Both mean square and almost sure convergence of the input sequence to the desired input are strictly established for noise-free systems and stochastic systems, respectively, where the measurement output suffers from random data dropouts. Illustrative simulations are provided to verify the theoretical results.

KEYWORDS

almost sure convergence, Bernoulli model, data dropout, iterative learning control, Markov chain, mean square convergence, stochastic sequence

1 | INTRODUCTION

Iterative learning control (ILC) is a branch of intelligent control as it can improve tracking performance whenever a given tracking task is completed repeatedly. In such a case, the tracking information and the corresponding input signal in previous iterations/cycles/batches are used to construct the input signal for the current iteration/cycle/batch from which a learning mechanism is introduced to ensure asymptotical convergence along the iteration axis. As a consequence, ILC is much suitable for systems that can complete a given task in a finite time interval and repeat it successively. Since its introduction in the work of Arimoto et al[1] in 1984 for robot control, ILC has gained a lot of developments both in theory and applications over the past 3 decades.[2-4] Many related topics have been studied such as robust ILC,[5] distributed ILC,[6-8] monotonic convergence,[9] interval ILC,[10] and initial resetting condition,[11] among others.

As fast developments of communication and network techniques, many systems have adopted the networked control structure, that is, the controller and the plant in such systems are separated in different sites and communicate with each other through wired/wireless networks. For example, when considering an application of ILC to the robot fish in the laboratory,[12] the control algorithm is run on a computer, and the computer is communicated with the robot fish through a wireless network for data and command transmission. Similar implementation goes to the unmanned aerial vehicle routine surveillance control, where the control center for updating the control signals and the unmanned aerial vehicles for continuous cruising are separated and communicate through wireless networks. Moreover, in the studies of distributed ILC,[6-8,13] the communication of different agents is also through wireless networks. Therefore, a natural and critical problem is the data dropout, which damages the tracking performance. This problem motivates us to consider the design and analysis of ILC in the presence of random data dropouts.

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Some earlier attempts have been reported.[14-27] In the next section, we will give a brief literature review of related studies on ILC in the presence of data dropouts from 3 aspects, namely, data dropout model, data dropout position, and convergence meaning. From the literature review, we have observed several facts: (1) most papers adopt the classic Bernoulli model for describing data dropouts, (2) most papers assume that the data dropouts only occur at the measurement side, and (3) convergence meaning is scattered in mathematical expectation, mean square, and almost sure senses in different papers.

In this paper, we propose a new analysis framework for the ILC problem under random data dropout environments. In this framework, the random sequence model (RSM), the Bernoulli variable model (BVM), and the Markov chain model (MCM) for data dropouts are all taken into consideration. Moreover, both mean square convergence and almost sure convergence of the input sequence to the desired input are established, from which the convergence in mathematical expectation is a direct corollary. Furthermore, although we restrict our discussions to the case that data dropouts occur at the measurement side, the extension to the general case that data dropouts occur at both measurement and actuator sides is easy to establish without additional limitations on the successive data dropouts. In addition, while we consider the classic P-type algorithm in this paper to clarify our idea, the extensions to other types of ILC algorithms such as the PD type and the current-iteration-feedback-integrated type can be derived following similar steps. We should point out that this paper focuses on control over networks, which is distinctly different from papers concerning control of networks, such as those by Meng and Moore,[7] Xiong et al,[8] and Xiong et al.[13] By control over networks, we mean the control signal is transmitted through networks, whereas by control of networks, we mean the control is constructed for a multiagent system (consisting of several agents or subsystems). Thus, the 2 problems have different research concerns.

This paper is arranged as follows. Section 2 presents a brief literature review of the contributions on ILC in the presence of data dropouts. Section 3 provides the problem formulation including system formulation, data dropout models, and the control objective. The detailed convergence analysis under the new framework for linear systems without and with stochastic noise is elaborated in Sections 4 and 5. Illustrative simulations are given in Section 6. Section 7 concludes this paper.

Notation. $\mathfrak{R}$ is the set of real numbers, and $\mathfrak{R}^n$ is the n-dimensional space. $\mathbb{P}(\cdot)$ denotes the probability of its indicated event, and $\mathbb{E}$ denotes the mathematical expectation of its indicated random variable. I_n denotes the unit matrix with dimension $n\times n$. The subscript n may be omitted where no confusion exists. $\mathbf{0}_{m\times n}$ denotes the zero matrix with dimension $m\times n$, and it is abbreviated as $\mathbf{0}_n$ when $n = m$. The superscript T is used to denote the transpose of a vector or a matrix. For a vector x, $\|x\|^2 = x^Tx$ denotes the Euclidean norm with $\|x\| = \sqrt{x^Tx}$, and $\|x\|_M = x^TMx$ denotes a weighted norm with respect to a positive definite matrix M.

2 | LITERATURE REVIEW

In this section, we give a brief literature review on ILC for systems with random data dropouts and classify the contributions of the existing papers from 3 aspects, namely, random data dropout models, data dropout positions, and convergence meaning. From these 3 dimensions, we can get a comprehensive picture view of the state of the art.

2.1 | Data dropout models

There are only 2 states describing the transmission: successful transmission and loss. Thus, if we introduce a random variable to describe the data dropouts, it is a binary variable. Usually, we let the variable be 1 if the corresponding data packet is successfully transmitted through the wired/wireless networks; we let the variable be 0 otherwise. Moreover, such variable is inherently random, and thus, we should introduce some additional model for the binary variable to give a characterization of the randomness of data dropouts.

The most popular model for the data dropout should go to the Bernoulli model. In this model, the random variable takes the value of 1 with success probability p and the value of 0 with failure probability $q = 1 - p$. Moreover, the data dropouts for different packets occur independently. In other words, this model has a clear probability distribution and good independence. Therefore, it is widely used in many papers addressing the data dropout topic. Most ILC papers also adopted this model[14-27] with/without extra requirements on data dropouts.

There are a few ILC papers dropping this model. Pan et al[28] gave an elaborate investigation of the effect of data dropouts. Thus, the authors mainly considered the case that only a single packet was lost during the transmission and provided a specific derivation for the effect on the input error and tracking performance. As to the multiple-packet-loss case, a general discussion was given instead of strict analysis and description. Specifically, the authors claimed that the data dropout level should be far smaller than 100% to ensure a satisfactory tracking performance.

The works of Shen and Wang[29,30] provided a so-called RSM for data dropouts. Specifically, the sequence of the data dropout variables along the iteration axis was not assumed to be with any specific probability distribution. In other words, the statistical property of the data dropouts can vary along the iteration axis. Thus, the steady distribution in the Bernoulli model is removed. However, to ensure asymptotical convergence of the input sequence, an additional requirement was imposed to the data dropout model in the works of Shen and Wang[29,30]: that there should exist a sufficient large number K such that during any successive K iterations, at least one data dropout variable takes the value of 1. In other words, the data should be successfully transmitted from time to time.

There is another model for data dropouts, ie, the MCM, which has been used in some papers addressing other control strategies. In this model, the data dropouts have some dependence on the previous event. That is, the loss or not of the current packet would affect the probability of successful transmission for the next packet. In the ILC under data dropouts, this model has not been discussed.

2.2 | Data dropout positions

In the networked ILC, the plant and the learning controller are separated in different sites and communicate with each other through wired/wireless networks. Thus, there are 2 channels connecting the plant and the learning controller. One channel is at the measurement side to transmit the measured output information back to the learning controller. The other channel is at the actuator side to transmit the generated input signal to the plant so that the operation process can continuously run.

When considering the data dropout problem for ILC, the position at which data dropout occurs is usually assumed to be the measurement side. In other words, only the network at the measurement side is assumed to be lossy, and the network at the actuator side is assumed to work well in most papers.[14-17,19,20,23-26,29,30] In these papers, the generated input signal can be always sent to the plant without any loss. Although some papers claimed that their results can be extended to the general case where the networks at the measurement and actuator sides suffered random data dropouts, it is actually not a trivial extension.

Specifically, when the network at the measurement side suffers random data dropouts, the output signal of the plant may or may not be successfully transmitted. One simple mechanism for treating the measured data is as follows: if the measured output is successfully transmitted, then the learning controller would employ such information for updating; if the measured output is lost during transmission, then the learning controller would stop updating until the corresponding output information is successfully transmitted. One may find that the lost data are simply replaced by 0 in this mechanism. For the case that data dropout occurs only at the measurement side, such simple mechanism is sufficient to ensure the learning process as long as the network is not completely broken down. However, when considering the data dropout at the actuator side, it is clear that the lost input signal cannot be simply replaced by 0 as it would greatly damage the tracking performance. That is, if the network at the actuator side suffers data dropouts, the lost input signal must be compensated with a suitable packet to maintain the operation process of the plant. This observation motivates the investigation on compensation mechanisms for the lost data.[18,21,22,28]

Pan et al[28] gave an earlier attempt on compensating the lost data. When 1 packet of the input signal is lost at the actuator side, the one-time-instant ahead input signal is applied to compensate for the lost one. That is, if the input at time instant t is lost, it would be compensated with the input at time instant $t-1$. When 1 packet of the output signal is lost at the measurement side, a similar compensation mechanism is applied. It is worth noting that the data dropouts at the measurement side and the actuator side are separately discussed in the work of Pan et al.[28] Moreover, this mechanism was then adopted by Bu et al[18] for a Bernoulli model of random data dropouts occurring at both the measurement and actuator sides simultaneously. We should emphasize that, as a natural consequence, the data at adjacent time instants at the same iteration cannot be dropped simultaneously due to the inherent compensation requirement. Another compensation mechanism is to apply the corresponding data from the last iteration as shown in the works of Huang and Fang[21] and Liu and Ruan.[22] That is, if the data packet at the kth iteration is lost during transmission, it is compensated with the packet at the $(k-1)$th iteration with the same time instant label. In such an assumption, the successive data dropouts along the time axis are allowed; however, it restricts that there was no simultaneous data dropout at the same time instant across any 2 adjacent iterations. In other words, no successive data dropouts along the iteration axis are allowed.

In short, the contributions in the aforementioned works[18,21,22,28] show that the newly introduced compensation mechanisms impose additional limitations to the data dropout models. In fact, the inherent difficulty of convergence analysis lies in the asynchronism between the computed input of the learning controller and the actual input fed to the plant. A recent paper[27] solved this problem according to the Bernoulli model allowing successive data dropouts along both time and iteration axes and provided a simple compensation mechanism with the iteration-latest available packet.

2.3 | Convergence meaning

In this subsection, we review the analysis techniques and the related convergence results, particularly the convergence meaning in considering the randomness of data dropouts apart from optional stochastic noise.

Ahn et al provided earlier attempts on ILC for linear systems in the presence of data dropouts.[14-16] The Kalman filtering–based technique, which was first proposed by Saab,[31] was applied, and thus, the mean square convergence of the input sequence was obtained. The main difference among the aforementioned papers[14-16] lies in the position where data dropouts occur. Specifically, in the first paper,[14] the output vector was assumed to be lossy; in the second paper,[15] this assumption was relaxed to the case where only partial dimensions of the output may suffer data dropouts; and in the third paper,[16] the data dropouts at both the measurement and actuator sides were taken into account. In short, the Kalman filtering–based technique was deeply investigated in the series of works by Ahn et al.

Bu et al[17-20] gave different angles to solve this problem. In the first paper,[17] the exponential stable result of asynchronous dynamical systems[49] was referred to establish the convergence condition of ILC under data dropouts. As a result, the randomness of data dropouts was not involved in the analysis steps. In the second paper,[18] such randomness was eliminated from the recursion by taking mathematical expectation; thus, the algorithm was converted into a deterministic type, and then, the design and analysis of the convergence followed the conventional way. Therefore, the convergence was clearly in the mathematical expectation sense. In the third paper,[19] a new H-∞ framework was defined along the iteration axis, and then, the related control problem was solved in the newly defined framework. That is, the kernel objective was to satisfy an H-∞ performance index in the mean square sense. A linear matrix inequality design condition for the learning gain matrix was also provided. In the fourth paper,[20] the widely used 2-dimensional system approach was revisited to deal with data dropouts. A mean square asymptotically stable result was obtained, and the design condition for the learning gain matrix was solved through linear matrix inequality techniques. In short, the evolution dynamics along the iteration axis was carefully studied, and related techniques are applied for the design and analysis of ILC.

There are some other scattered results on this topic.[21-23,28] Pan et al[28] proposed a detailed analysis of the effect of packet loss for the sampled ILC. Specifically, a single packet loss at the measurement side and the actuator side was evaluated separately to study the inherent influence of data dropout on the tracking performance. In other words, a deterministic analysis was given according to the input error. The results in the work of Pan et al[28] revealed that neither contraction nor expansion occurred for the input error if the corresponding packet was lost during transmission. Such a technique was further exploited and used in the work of Huang and Fang[21] to study the general data dropout case. In the work of Liu and Ruan,[22] a mathematical expectation was taken to the recursive inequality of input error to eliminate the randomness of data dropouts similar to the work of Bu et al,[18] and then, the conventional contraction mapping method was used to derive the convergence results. Moreover, to construct explicit contraction mapping, the conditions in the work of Liu and Ruan[22] were much conservative, and it may be further relaxed. Similar techniques were also used in the work of Liu and Xu,[23] an incorporation with the conventional α-norm technique to derive convergence in the mathematical expectation sense.

Shen et al mainly contributed the almost sure convergence results of ILC under data dropout environments. In the work of Shen and Wang,[24] a simple case that the whole iteration was packed and transmitted as a single packet was investigated by a switched system approach. Specifically, the evolution along the iteration axis was formulated as a switched system, and the statistical properties were recursively computed. Then, the convergence in the sense of expectation, mean square, and almost sure was established in turn. In the work of Shen and Wang,[29] based on stochastic approximation theory, the almost sure convergence of the input sequence was proved for the case that the data dropouts were modeled by an RSM. This result was then extended to the unknown control direction case in the work of Shen and Wang.[30] For the traditional Bernoulli model of data dropouts, the essential difficulty in obtaining the almost sure convergence lies in the random successive data dropouts along the iteration axis. This problem was solved in the works of Shen et al[25,26] for linear and nonlinear stochastic systems, respectively. The authors of these papers proceeded to investigate the general data dropouts at both measurement and actuator sides without any additional requirements but the Bernoulli assumption in the work of Shen and Xu.[27] When data dropouts occur at the actuator sides, there is a newly introduced asynchronism between the computed control generated by the learning controller and the actual control

TABLE 1 Classification of the papers on iterative learning control under data dropouts

Refs	Model			Position		Convergence			
	RSM	BVM	MCM	Measurement	Actuator	ME	MS	AS	DA
Ahn et al[14]		√		√			√		
Ahn et al[15]		√		√			√		
Ahn et al[16]		√		√	√		√		
Bu et al[20]		√		√			√		
Bu and Hou[17]		√		√					√
Bu et al[18]		√		√	√	√			
Bu et al[19]		√		√			√		
Huang and Fang[21]		√		√	√				√
Liu and Ruan[22]		√		√	√	√			
Liu and Xu[23]		√		√		√			
Pan et al[28]		√		√	√				√
Shen and Wang[24]		√		√		√	√	√	
Shen and Wang[29]	√			√				√	
Shen and Wang[30]	√			√				√	
Shen and Xu[27]		√		√	√		√	√	
Shen et al[25]		√		√				√	
Shen et al[26]		√		√				√	

Abbreviations: AS, almost sure; BVM, Bernoulli variable model; DA, deterministic analysis; MCM, Markov chain model; ME, mathematical expectation; MS, mean square; RSM, random sequence model.

fed to the plant. Such asynchronism was characterized by a Markov chain in the aforementioned work,[27] and then, the mean square and almost sure convergence was established.

2.4 | Further remarks

The recent progress on ILC in the presence of data dropouts is classified in Table 1 according to the data dropout model, data dropout position, and convergence meaning. From this Table, we have observed several points.

- In most papers, the data dropout is modeled by the Bernoulli random variable, while the results according to the RSM are rather limited. Moreover, for the MCM, no result has been reported.
- All the papers consider the data dropout occurring at the measurement side, and only a few papers address the case at the actuator side. As we have previously explained, the latter case would involve an essential influence on the controller design and convergence analysis.
- The convergence meaning is scattered in different papers. Mean square and almost sure convergence implies the convergence in the mathematical expectation sense. However, they cannot imply each other according to the probability theory. Thus, it is of interest to propose an in-depth framework for the design and analysis of ILC in both senses simultaneously.

Based on this progress, we will propose a comprehensive framework for the convergence analysis of ILC under various data dropout models. In contrast to the current status, we have the following highlights. First of all, the new framework is applicable to all the proposed models of data dropouts, and thus, the blank for the MCM is filled (differing from almost all relevant papers). Moreover, our method can be extended to the actuator-side case without imposing further restrictions on the successive data dropouts (differing from the one-side data dropout papers[17,19,20,25,26,29,30] and restricted two-side data dropout papers[18,21,22,28]). Furthermore, we will reveal the essential connection between the convergence results in the mean square and almost sure sense and then establish the convergence results for both noise-free and noised systems, respectively (differing from the mathematical expectation–based convergence papers[17-20]). In short, the ILC problem under data dropouts is deeply discussed and resolved in this paper.

3 PROBLEM FORMULATION

In this section, we will formulate the system, models for data dropouts, and the control objective in turn.

3.1 System formulation

Consider the following linear time-varying system:

$$x_k(t+1) = A_t x_k(t) + B_t u_k(t) + w_k(t+1),$$
$$y_k(t) = C_t x_k(t) + v_k(t), \tag{1}$$

where k is the iteration number, $k = 1, 2, \cdots$, t is the time instant, $t = 0, 1, \cdots, N$, and N is the iteration length. The variables $x_k(t) \in \mathfrak{R}^n$, $u_k(t) \in \mathfrak{R}^p$, and $y_k(t) \in \mathfrak{R}^q$ are the system state, input, and output, respectively. The notations $w_k(t) \in \mathfrak{R}^n$ and $v_k(t) \in \mathfrak{R}^q$ are the system and measurement noise, respectively. In addition, A_t, B_t, and C_t are system matrices with appropriate dimensions.

If the stochastic noise values $w_k(t)$ and $v_k(t)$ are absent, ie, $w_k(t) = v_k(t) = 0, \forall k, t$, we term the system a noise-free system. Otherwise, if variables $w_k(t)$ and $v_k(t)$ are described by random variables, we term the system a stochastic system.

In this paper, we assume that the system relative degree is τ, $\tau \geq 1$, that is, for any $t \geq \tau$, we have

$$C_t A_{t+1-i}^{t-1} B_{t-i} = 0, \quad 1 \leq i \leq \tau - 1, \tag{2}$$

$$C_t A_{t+1-\tau}^{t-1} B_{t-\tau} \neq 0, \tag{3}$$

where $A_i^j \triangleq A_j A_{j-1} \cdots A_i, j \geq i$, and $A_{i+1}^i \triangleq I_n$.

Remark 1. The relative degree implies the smallest structure delay of the input effect on its corresponding output. For example, if the relative degree $\tau = 1$, the input at time instant t would have an effect on the output at time instant $t+1$ but no effect on the output at time instant t. The relative degree is an intrinsic property of the system and, thus, is usually time invariant. Moreover, assuming the relative degree to be τ and starting the operation from the time instant $t = 0$, we find that the first controllable output appears at time instant $t = \tau$, which is driven by $u_k(0)$. In other words, the outputs at time $t = 0$ up to $t = \tau - 1$ are uncontrollable in such a situation. As a consequence, these outputs would be formulated in the initialization condition. In addition, considering the MIMO system formulation, the relative degree may vary for different dimensions of the output vector, that is, different dimensions of the output vector have different relative degree values. It is straightforward to extend the following derivations to this case. Therefore, we omit the tedious extensions to make a concise layout.

Denote the desired reference as $y_d(t), t \in \{0, 1, \ldots, N\}$. Without loss of generality, we assume that the reference is achievable, that is, with a suitable initial value of $x_d(0)$, there exists a unique input $u_d(t)$ such that

$$x_d(t+1) = A_t x_d(t) + B_t u_d(t),$$
$$y_d(t) = C_t x_d(t). \tag{4}$$

Denote the tracking error as $e_k(t) \triangleq y_d(t) - y_k(t), t \in \{0, 1, \ldots, N\}$.

Remark 2. Note that the system relative degree is τ, implying that the output at time instant $t = 0$ up to $t = \tau - 1$ cannot be affected by the input. Therefore, the actual tracking reference is $y_d(t), \tau \leq t \leq N$, whereas the initial τ outputs from $t = 0$ up to $\tau - 1$ are regulated by the initialization condition. Moreover, the uniqueness of the desired input $u_d(t)$ can be guaranteed if the matrix $C_t A_{t+1-\tau}^{t-1} B_{t-\tau}$ is of full-column rank. That is, the input $u_d(t)$ can be recursively computed from the nominal model (4) for $t \geq \tau$ as follows:

$$u_d(t-\tau) = \left[\left(C_t A_{t+1-\tau}^{t-1} B_{t-\tau} \right)^T \left(C_t A_{t+1-\tau}^{t-1} B_{t-\tau} \right) \right]^{-1} \left(C_t A_{t+1-\tau}^{t-1} B_{t-\tau} \right)^T \times \left(y_d(t) - C_t A_{t-\tau}^{t-1} x_d(t-\tau) \right). \tag{5}$$

The special case of Equation 5, with τ being 1, has been explicitly given in many existing papers.[31,32] It should be emphasized that the full-column rank requirement is not strict as it has been proved necessary for perfect tracking.[33,34] As a consequence, formulation (4) is a mild assumption for the system, which has been used in many existing ILC papers. When the coupling matrix is of full-row rank rather than full-column rank, which usually implies that the dimension of the input is greater than that of the output, it is found that only the asymptotical convergence of the tracking error is ensured in many papers (see, eg, the work of Saab[35]). Moreover, recent papers[36,37] have extended the rank conditions of coupling matrices from an iteration-invariant case to an iteration-varying case, which is a promising issue in handling nonrepetitive uncertainties.

The following mild assumptions are given for system (1).

Assumption 1. The system initial value satisfies that $x_k(0) = x_d(0)$, where $x_d(0)$ is consistent with the desired reference $y_d(0)$ in the sense that $y_d(0) = C_0 x_d(0)$.

Remark 3. This initialization condition is critical for ensuring the accurate tracking performance of the whole iteration and, thus, is an important issue in the ILC field. Assumption 1 is the well-known identical initialization condition. This condition is a basic requirement for time and space resetting of the system operation and, thus, is widely used in most ILC papers. Moreover, many scholars have contributed to relaxing this condition by introducing initial rectifying or learning mechanisms; however, either additional system information or tracking information is required when using the initial learning mechanisms.[38,39] Note that the focus of this paper is in proposing the comprehensive analysis of ILC under data dropout environments; thus, we use Assumption 1 to make the paper concentrated.

Define the σ-algebra $\mathcal{F}_k = \sigma\big(x_i(t), u_i(t), y_i(t), w_i(t), v_i(t), 1 \le i \le k, 0 \le t \le N\big)$ (ie, the set of all events induced by these random variables) for $k \ge 1$.

Assumption 2. The stochastic noise variables $\{w_k(t)\}$ and $\{v_k(t)\}$ are martingale difference sequences along the iteration axis with finite conditional second moments. That is, for $t \in \{0, 1, \cdots, N\}$, $\mathbb{E}\{w_{k+1}(t)|\mathcal{F}_k\} = 0$, $\sup_k \mathbb{E}\{\|w_{k+1}(t)\|^2|\mathcal{F}_k\} < \infty$, $\mathbb{E}\{v_{k+1}(t)|\mathcal{F}_k\} = 0$, $\sup_k \mathbb{E}\{\|v_{k+1}(t)\|^2|\mathcal{F}_k\} < \infty$.

Remark 4. The system for which the ILC method is applicable should be repeated so that the tracking performance can be gradually improved along the iteration axis. Consequently, the stochastic noise variables are usually independent along the iteration axis, from which Assumption 2 is mild and widely satisfied in practical applications. It is evident that the classical zero-mean white noise satisfies this assumption.

To facilitate the analysis in the following sections, we give the lifting forms of system (1). To this end, define the super-vectors as follows:

$$U_k = \left[u_k^T(0), u_k^T(1), \cdots, u_k^T(N-\tau)\right]^T, \tag{6}$$

$$Y_k = \left[y_k^T(\tau), y_k^T(\tau+1), \cdots, y_k^T(N)\right]^T. \tag{7}$$

Similarly, U_d and Y_d can be defined by replacing the subscript k in the above equations with d. The associated transfer matrix $\mathbf{H}$ can be formulated as

$$\mathbf{H} = \begin{bmatrix} C_\tau A_1^{\tau-1} B_0 & \mathbf{0}_{q\times p} & \cdots & \mathbf{0}_{q\times p} \\ C_{\tau+1} A_1^{\tau} B_0 & C_{\tau+1} A_2^{\tau} B_1 & \cdots & \mathbf{0}_{q\times p} \\ \vdots & \vdots & \ddots & \vdots \\ C_N A_1^{N-1} B_0 & C_N A_2^{N-1} B_1 & \cdots & C_N A_{N-\tau+1}^{N-1} B_{N-\tau} \end{bmatrix}. \tag{8}$$

Therefore, we have the following relationship between the input and the output:

$$Y_k = \mathbf{H} U_k + M x_k(0) + \xi_k \tag{9}$$

and

$$Y_d = \mathbf{H} U_d + M x_d(0), \tag{10}$$

where $M = [(C_\tau A_0^{\tau-1})^T, \cdots, (C_N A_0^{N-1})^T]^T$, and

$$\xi_k = \left[\left(\sum_{i=1}^{\tau} C_\tau A_i^{\tau-1} w_k(i) + v_k(\tau)\right)^T, \left(\sum_{i=1}^{\tau+1} C_{\tau+1} A_i^{\tau} w_k(i) + v_k(\tau+1)\right)^T, \cdots, \left(\sum_{i=1}^{N} C_N A_i^{N-1} w_k(i) + v_k(N)\right)^T\right]^T. \tag{11}$$

Recalling the tracking error $e_k(t) = y_d(t) - y_k(t)$, we denote the lifted tracking error $E_k \triangleq Y_d - Y_k$. Then, it is evident that

$$E_k = Y_d - Y_k = \mathbf{H}(U_d - U_k) - \xi_k, \tag{12}$$

where Assumption 1 is applied. These formulations will be used in the convergence analysis only.

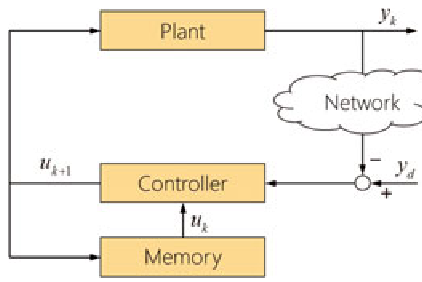


FIGURE 1 Block diagram of the networked iterative learning control [Colour figure can be viewed at wileyonlinelibrary.com]

3.2 | Models for data dropouts

In this subsection, we give the 3 common models of data dropouts and detail the differences among the models. Prior to the formulation, we first present the networked structure of ILC.

The block diagram of the networked ILC considered in this paper is given in Figure 1, in which, without loss of generality, only the network at the measurement side is assumed to be lossy, whereas the network at the actuator side is assumed to work well. The extension to the case that the networks at both sides suffer from the data dropout problem can be derived following the similar steps in this paper (cf Remark 10). Therefore, to make our idea easy to follow, we restrict our discussions to the one-side data dropout case.

The data dropout occurring or not, which could be regarded as a switch that opens and closes the network in a random manner, is denoted by a random variable $\gamma_k(t)$. Therefore, there are 2 possible states of the variable $\gamma_k(t)$. Specifically, we let $\gamma_k(t)$ be equal to 1 if the corresponding tracking error $y_k(t)$ is successfully transmitted and to 0 otherwise. In this paper, without loss of generality, the information for each time instant is packed as a data packet and transmitted, that is, $y_k(t)$ also denotes an individual data packet containing the output information at time instant t for the kth iteration.

In this paper, we consider the following 3 most common models of data dropouts.

- Random sequence model (RSM): For each t, the measurement packet loss is random without obeying any certain probability distribution, but there is a positive integer $K \geq 1$ such that, at least in 1 iteration, the measurement is successfully sent back during the successive K iterations.
- Bernoulli variable model (BVM): The random variable $\gamma_k(t)$ is independent for different values of time instant t and iteration number k. Moreover, $\gamma_k(t)$ obeys a Bernoulli distribution with
$$\mathbb{P}(\gamma_k(t) = 1) = \bar{\gamma}, \quad \mathbb{P}(\gamma_k(t) = 0) = 1 - \bar{\gamma}, \tag{13}$$
where $\bar{\gamma} = \mathbb{E}\gamma_k(t)$ with $0 < \bar{\gamma} < 1$.
- Markov chain model (MCM): The random variable $\gamma_k(t)$ is independent for different values of time instant t. Moreover, for each t, the evolution of $\gamma_k(t)$ along the iteration axis follows a 2-state Markov chain, of which the probability transition matrix is
$$P = \begin{bmatrix} P_{11} & P_{10} \\ P_{01} & P_{00} \end{bmatrix} = \begin{bmatrix} \mu & 1-\mu \\ 1-\nu & \nu \end{bmatrix} \tag{14}$$
with $0 < \mu, \nu < 1$, where $P_{11} = \mathbb{P}(\gamma_{k+1}(t) = 1|\gamma_k(t) = 1)$, $P_{10} = \mathbb{P}(\gamma_{k+1}(t) = 0|\gamma_k(t) = 1)$, $P_{01} = \mathbb{P}(\gamma_{k+1}(t) = 1|\gamma_k(t) = 0)$, and $P_{00} = \mathbb{P}(\gamma_{k+1}(t) = 0|\gamma_k(t) = 0)$.

Remark 5. The RSM is illustrated in Figure 2 where a horizontal bar denotes an iteration process. In any bar, the white rectangle and the black rectangle denote the lost packet and the successfully transmitted packet, respectively. The gray part of each horizontal bar denotes the omission part. The RSM implies that, for an arbitrary time instant t, the corresponding output information can be received at least once for any successive K iterations. As shown in Figure 2, taking the time instant $t = 4$ for example, there is at least 1 black rectangle for any successive K horizontal bars. Moreover, this model can be formulated using the random variable $\gamma_k(t)$ as follows: for each t, $\sum_{i=0}^{K-1} \gamma_{k+i}(t) \geq 1$ for all $k \geq 1$. It is worth pointing out that we only require the existence of the number K rather than its specific value, that is, the number K is not necessary to be known prior, and it is not involved in the design of the ILC update law later. In fact, this model means that the output information should not be lost too much to ensure the learning ability in a somewhat deterministic point of view.

Remark 6. The number K of the RSM indicates that the maximum length of successive data dropouts is $K - 1$. Thus, the case $K = 1$ means no data dropout occurring, whereas the case $K = 2$ means no successive data dropout occurring for any

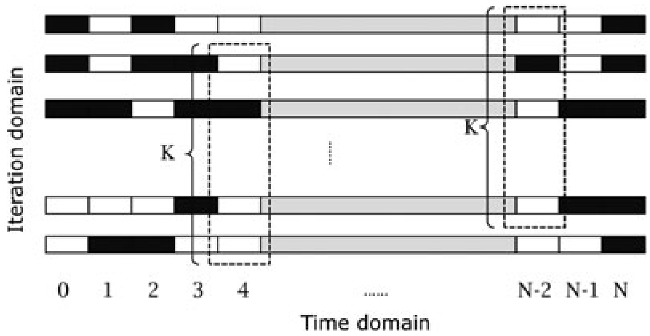


FIGURE 2 Illustration of the random sequence model

2 subsequent iterations. Moreover, the value of the successive iteration number K is a reflection of the rate of data dropouts. However, it is not equivalent to the data dropout rate (DDR), which can be formulated as $\lim_{n\to\infty} 1/n \times \left[\sum_{k=1}^{n}\left(1-\gamma_k(t)\right)\right]$. In fact, DDR denotes the average level of data dropouts along the iteration axis, whereas K implies the worst case of successive data dropouts. In other words, a larger K value usually corresponds to a higher DDR, whereas a smaller K value usually corresponds to a lower DDR. However, the connection between K and DDR need not necessarily be positively related.

Remark 7. The mathematical expectation $\bar{\gamma}$ of the BVM is closely related to the DDR in light of the law of large numbers, that is, DDR is equal to $1-\bar{\gamma}$. Specifically, the data dropout is independent along the iteration axis; thus, $\lim_{n\to\infty} 1/n \times \left[\sum_{k=1}^{n}\left(1-\gamma_k(t)\right)\right] = 1-\mathbb{E}\gamma_k(t) = 1-\bar{\gamma}$. If $\bar{\gamma}=0$, implying that the network is completely broken down, then no information can be received from the plant, and thus, no algorithm can be applied to improve the tracking performance. If $\bar{\gamma}=1$, implying that no data dropout occurs, then the framework converts into the classical ILC problem. In this paper, with a framework for designing and analyzing the ILC update law under data dropouts, we simply assume $0<\bar{\gamma}<1$. Moreover, the statistics property of $\gamma_k(t)$ is assumed to be identical for different time instants for a concise expression. The extension to the time-dependent case, ie, the case that $\mathbb{E}\gamma_k(t)=\bar{\gamma}_t$, is straightforward without additional efforts.

Remark 8. The MCM is general for modeling the data dropouts. The transition probabilities μ and ν denote the average level of retaining the same state for successful transmission and loss, respectively. If $\mu+\nu=1$, then the MCM converts into the BVM. That is, the BVM is a special case of the MCM. It is worth pointing out that all 3 models are widely investigated in the field of networked control systems, such as in the works of Lin and Antsaklis[40] for the RSM, Sinopoli et al[41] for the BVM, and Shi and Yu[42] for the MCM.

Remark 9. In this remark, we comment the differences among the 3 models. The RSM differs from both the BVM and the MCM as it requires no probability distribution or statistics property of the random variable $\gamma_k(t)$. However, the RSM pays the price that the successive data dropout length is bounded, compared with BVM and MCM. Specifically, both BVM and MCM admit arbitrary successive data dropouts associated with a suitable occurring probability. Consequently, the RSM cannot cover BVM/MCM, and vice versa. It should be pointed out that the RSM implies that the data dropout is not totally stochastic. Moreover, the difference between the BVM and the MCM lies in the point that the data dropout occurs independently along the iteration axis for the BVM, while dependently for the MCM. The independence of data dropout admits some specific computations such as mean and variance (compare with the work of Shen et al[43]) and then derives the convergence analysis. Such a technique is not applicable for the MCM.

Remark 10. When the network at the actuator side is lossy, a simple updating mechanism for the input fed to the plant is the holding strategy. That is, if the newly generated input signal is successfully transmitted, the input fed to the plant is updated; if the newly generated input signal is lost during transmission, the input fed to the plant retains the last available value. Using this updating mechanism, the following convergence analysis can be extended to the general data dropout case. Specifically, when data dropout occurring at the actuator side, it is seen that the input signal generated by the controller and the one fed to the plant are not always identical. Such asynchronism between the 2 input signals can be analyzed following similar steps as in the work of Shen and Xu[27] and shown to be bounded (for the RSM model) or Markovian (for BVM and MCM models). Thus, the analysis techniques proposed in this paper can be applied.

3.3 | Control objective

The conventional control objective of ILC for a noise-free system is to construct an update law such that the generated input sequence can guarantee the asymptotical precise tracking to the desired reference, that is, the output $y_k(t)$ can track the given trajectory $y_d(t)$ asymptotically for the specified time instants. However, when dealing with stochastic systems, it is impossible to achieve this control objective because of the existence of the unpredictable stochastic noise variables $w_k(t)$ and $v_k(t)$. That is, we cannot expect that $y_k(t) \to y_d(t)$, $\forall t$, for stochastic systems, as the iterations increase to infinity. Note that the stochastic noise variables cannot be eliminated by any algorithm in advance; thus, the best achievable control objective should ensure that the desired reference can be precisely tracked by the output with removing these stochastic noise variables. To this end, the control objective in this paper is to design an ILC algorithm guaranteeing the precise tracking of the input rather than the output, that is, $u_k(t) \to u_d(t)$ as $k \to \infty$, $t = 0, \cdots, N-\tau$. In fact, if we can guarantee that $u_k(t) \to u_d(t)$, then the following averaged index of tracking errors is minimized:

$$V_t = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \|e_k(t)\|^2 = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \|y_d(t) - y_k(t)\|^2, \quad t \geq \tau.$$

In addition, if the stochastic noise variables are absent, then the precise convergence of the input guarantees the precise convergence of the output.

Moreover, when considering the stochastic systems, it is clear that all the inputs, states, and outputs are random variables. Even if the stochastic noise variables are removed, the random data dropouts also result in that the inputs, states, and outputs are random variables. Therefore, we should clarify the convergence meaning from the viewpoint of probability theory. Specifically, we have the following 3 types of convergence.

- Convergence in mathematical expectation: the input sequence $\{u_k(t)\}$ is called to achieve convergence in mathematical expectation if $\lim_{k\to\infty} \mathbb{E} u_k(t) = u_d(t)$ for $t = 0, \cdots, N-\tau$.
- Mean square convergence: the input sequence $\{u_k(t)\}$ is called to achieve mean square convergence if $\lim_{k\to\infty} \mathbb{E}\|u_k(t) - u_d(t)\|^2 = 0$ for $t = 0, \cdots, N-\tau$.
- Almost sure convergence: the input sequence $\{u_k(t)\}$ is called to achieve almost sure convergence if $\lim_{k\to\infty} u_k(t) = u_d(t)$ with probability 1 for $t = 0, \cdots, N-\tau$.

As is well known, both mean square convergence and almost sure convergence imply the convergence in the mathematical expectation sense. Thus, if we establish either mean square convergence or almost sure convergence, then the convergence in mathematical expectation is a direct corollary. However, mean square convergence and almost sure convergence cannot imply each other generally. Therefore, in the rest of this paper, our analysis objective is to show the mean square convergence and almost sure convergence of the proposed ILC algorithms under all 3 data dropout models.

3.4 | Preliminaries

Lemma 1. *Let $\{\vartheta_k\}$ be a sequence of positive real numbers and such that*

$$\vartheta_{k+1} \leq (1 - d_1 a_k)\vartheta_k + d_2 a_k^2 (d_3 + \vartheta_k), \tag{15}$$

where $d_i > 0$, $i = 1, \cdots, 3$, are constants, and a_k satisfies $a_k > 0$, $\sum_{k=1}^{\infty} a_k = \infty$, and $\sum_{k=1}^{\infty} a_k^2 < \infty$, then $\lim_{k\to\infty} \vartheta_k = 0$.[27]

The proof of this lemma is put in the Appendix for smooth readability.

Lemma 2. *Let $X(n)$ and $Z(n)$ be nonnegative stochastic processes (with finite expectation) adapted to increasing σ-algebra $\{\mathcal{F}_n\}$ and such that*

$$\mathbb{E}\{X(n+1)|\mathcal{F}_n\} \leq X(n) + Z(n), \tag{16}$$

$$\sum_{n=1}^{\infty} \mathbb{E}[Z(n)] < \infty. \tag{17}$$

Then, $X(n)$ converges almost surely, as $n \to \infty$.[44]

4 | CONVERGENCE OF THE NOISE-FREE LINEAR SYSTEM

In this section, we consider the case that the stochastic noise variables are absent in Equation 1, that is, we consider the noise-free system, ie,

$$x_k(t+1) = A_t x_k(t) + B_t u_k(t),$$
$$y_k(t) = C_t x_k(t). \tag{18}$$

For such a system, the randomness is only resulted from the data dropouts, which provides us a concise view to address the influences of data dropouts and stochastic noise.

The P-type ILC update law is designed as follows:

$$u_{k+1}(t) = u_k(t) + \sigma\gamma_k(t+\tau)L_t e_k(t+\tau), \tag{19}$$

for $t = 0, \ldots, N-\tau$, where σ is a positive constant to be specified later, and $L_t \in \mathfrak{R}^{p\times q}$ is the learning gain matrix for regulating the control direction.

Remark 11. First, we emphasize again that the ILC update law is not limited to the classical P-type law, although we mainly focus on such type in this paper to make a concise expression. Second, it is evident that the design of the positive constant σ can be blended into the design of L_t. However, here, we provide the separated design procedure to elaborate on a clear design principle in the following analysis of this section as well as to provide a comparison with the design for the stochastic system case in the next section.

Now, lift the input along the time axis as in Equation 6. The update law (19) can be rewritten as follows:

$$U_{k+1} = U_k + \sigma\Gamma_k \mathbf{L} E_k, \tag{20}$$

where $E_k = Y_d - Y_k$, defined as in Equation 12 with $\xi_k = 0$, is the stacked vector of the tracking errors for $t = \tau, \cdots, N$, and Γ_k and $\mathbf{L}$ are defined by

$$\Gamma_k = \begin{bmatrix} \gamma_k(\tau)I_q & & & \\ & \gamma_k(\tau+1)I_q & & \\ & & \cdots & \\ & & & \gamma_k(N)I_q \end{bmatrix}, \tag{21}$$

$$\mathbf{L} = \begin{bmatrix} L_0 & & & \\ & L_1 & & \\ & & \cdots & \\ & & & L_{N-\tau} \end{bmatrix}. \tag{22}$$

Clearly, $\Gamma_k = \text{diag}\{\gamma_k(\tau), \gamma_k(\tau+1), \cdots, \gamma_k(N)\} \otimes I_q$.

Recalling that $E_k = \mathbf{H}(U_d - U_k)$ and substituting this into Equation 20, we have

$$U_{k+1} = U_k + \sigma\Gamma_k \mathbf{L}\mathbf{H}(U_d - U_k). \tag{23}$$

We define $\Lambda_k \triangleq \Gamma_k \mathbf{L}\mathbf{H}$ and

$$\mathbf{L}\mathbf{H} = \begin{bmatrix} L_0 C_\tau A_1^{\tau-1} B_0 & \mathbf{0}_p & \cdots & \mathbf{0}_p \\ L_1 C_{\tau+1} A_1^{\tau} B_0 & L_1 C_{\tau+1} A_2^{\tau} B_1 & \cdots & \mathbf{0}_p \\ \vdots & \vdots & \ddots & \vdots \\ L_{N-\tau} C_N A_1^{N-1} B_0 & L_{N-\tau} C_N A_2^{N-1} B_1 & \cdots & L_{N-\tau} C_N A_{N-\tau+1}^{N-1} B_{N-\tau} \end{bmatrix}. \tag{24}$$

Since $\mathbf{LH}$ is a block lower triangular matrix, it is clear that the eigenvalue set of $\mathbf{LH}$ is a combination of the eigenvalue sets of $L_t C_{t+\tau} A_{t+1}^{t+\tau-1} B_t$, $t = 0, \cdots, N-\tau$. Moreover, Γ_k is a block diagonal matrix; thus, $\Lambda_k = \Gamma_k \mathbf{LH}$ is also a block lower triangular matrix with all eigenvalues being the eigenvalues of its diagonal blocks. Specifically, the eigenvalue of Λ_k is either equal to the eigenvalue of $\mathbf{LH}$ or equal to zero, depending on whether the corresponding variable $\gamma_k(t)$ is 1 or 0, respectively.

Note that each $\gamma_k(t)$ has 2 possible values, ie, 1 or 0, corresponding to that the data are successfully transmitted or not; thus, Γ_k has $\kappa \triangleq 2^{N+1-\tau}$ possible outcomes due to the independence of $\gamma_k(t)$ for different time instants. As a consequence, the newly defined $\Lambda_k = \Gamma_k \mathbf{LH}$ also has κ possible outcomes. Denote the set of all possible outcomes as $\mathfrak{S} = \{\Lambda^{(1)}, \cdots, \Lambda^{(\kappa)}\}$. Without loss of generality, we denote $\Lambda^{(1)} = \mathbf{LH}$ and $\Lambda^{(\kappa)} = \mathbf{0}_{(N+1-\tau)p}$, corresponding to the cases that all $\gamma_k(t)$ are equal to 1 and 0, respectively. The other $\kappa - 2$ alternatives are also block lower triangular matrices similar to $\mathbf{LH}$ but with one or more block rows of $\mathbf{LH}$ that are zero rows, corresponding to the time instants at which the packets are lost during transmission. In other words, for a matrix Λ_k, if the data packet at time instant t is lost during transmission, $t \geq \tau$, then the $(t+1-\tau)$th block row of Λ_k is a zero block row.

Now, we give the design condition of the learning gain matrix L_t, $0 \leq t \leq N-\tau$.

Learning gain matrix condition: In order to ensure the convergence of the P-type update law (19), the learning gain matrix L_t should satisfy that $-L_t C_{t+\tau} A_{t+1}^{t+\tau-1} B_t$ is a Hurwitz matrix, where a square matrix M is called Hurwitz if all the eigenvalues of M are with negative real parts.

Recalling the formulation of **LH** in Equation 24, we have that all eigenvalues of $-\mathbf{LH}$ are with negative real parts if $-L_t C_{t+\tau} A_{t+1}^{t+\tau-1} B_t$ is a Hurwitz matrix for $t = 0, \cdots, N-\tau$. By the Lyapunov theorem, for any negative definite matrix $\mathbf{S}$ with appropriate dimension, there is a positive definite matrix $\mathbf{Q}$ such that $-((\mathbf{LH})^T\mathbf{Q} + \mathbf{QLH}) = \mathbf{S}$. In the following, to facilitate the analysis, we let $\mathbf{S} = -I$. That is, there exists a positive matrix $\mathbf{Q}$ such that

$$(\mathbf{LH})^T\mathbf{Q} + \mathbf{QLH} = I. \tag{25}$$

Noting the difference between $\Lambda^{(i)}$ and $\mathbf{LH}$, we have

$$\left(\Lambda^{(i)}\right)^T\mathbf{Q} + \mathbf{Q}\Lambda^{(i)} \geq 0, \tag{26}$$

for $i = 2, \cdots, \kappa - 1$.

Define $\delta U_k \triangleq U_d - U_k$. Subtracting both sides of Equation 23 from U_d yields

$$\delta U_{k+1} = \left(I_{(N+1-\tau)p} - \sigma\Lambda_k\right)\delta U_k. \tag{27}$$

In the following subsections, we will give the zero-error convergence proofs of Equation 27 for 3 models in turn. To show the convergence, it is sufficient to establish the inherent contraction mapping of $I_{(N+1-\tau)p} - \sigma\Lambda_k$, in which Λ_k is a random matrix. For the RSM, the contraction cannot hold for each iteration, and thus, a technical lemma is first provided to obtain the joint contraction along the iteration axis. For the BVM and the MCM, the contraction is verified for each iteration based on the probability properties of the statistic models.

4.1 | RSM case

When considering the RSM case, no statistical property of the data dropout can be accessed and used; however, the bounded-length assumption of successive data dropouts ensures a somewhat deterministic way for convergence analysis.

To make a clear insight of the influence of the RSM case of data dropouts, we rewrite Equation 27 as follows:

$$\delta U_{k+K} = \left(I - \sigma\Lambda_{k+K-1}\right)\cdots\left(I - \sigma\Lambda_k\right)\delta U_k. \tag{28}$$

Denote

$$\Phi_{m,n} = \left(I - \sigma\Lambda_m\right)\cdots\left(I - \sigma\Lambda_n\right), \quad m \geq n. \tag{29}$$

Now, we give an estimate of $\Phi_{k+K-1,k}$ in the following lemma.

Lemma 3. *Consider the matrix product (29). If the learning matrix L_t satisfies that $-L_t C_{t+\tau} A_{t+1}^{t+\tau-1} B_t$ is a Hurwitz matrix and σ is small enough, then there exists a positive definite matrix* $\mathbf{Q}$ *such that*

$$\Phi_{k+K-1,k}^T\mathbf{Q}\Phi_{k+K-1,k} \leq \eta\mathbf{Q}, \quad 0 < \eta < 1, \quad \forall k. \tag{30}$$

Proof. As previously explained, all Λ_k are block lower triangular matrices; thus, the summation of Λ_k is also a block lower triangular matrix. In other words, $\sum_{i=0}^{K-1} \Lambda_{k+i}$ is a block lower triangular matrix. Moreover, the RSM assumption of data dropouts implies that all the diagonal blocks of $\sum_{i=0}^{K-1} \Lambda_{k+i}$ are with positive real parts in their eigenvalues for $k \geq 0$, which further implies that there exists some positive constant $c_1 > 0$ such that

$$\left(\sum_{i=0}^{K-1} \Lambda_{k+i}\right)^T\mathbf{Q} + \mathbf{Q}\left(\sum_{i=0}^{K-1} \Lambda_{k+i}\right) \geq c_1 I, \quad \forall k \geq 0. \tag{31}$$

Now, revisit the recursion of $\Phi_{k+K-1,k}$, and we have

$$\begin{aligned}\Phi^T_{k+K-1,k}\mathbf{Q}\Phi_{k+K-1,k} &= \left(I-\sigma\Lambda_k^T\right)\cdots\left(I-\sigma\Lambda_{k+K-1}^T\right)\mathbf{Q}\left(I-\sigma\Lambda_{k+K-1}\right)\cdots\left(I-\sigma\Lambda_k\right)\\ &\le \mathbf{Q}-\sigma\left[\left(\sum_{i=0}^{K-1}\Lambda_{k+i}\right)^T\mathbf{Q}+\mathbf{Q}\left(\sum_{i=0}^{K-1}\Lambda_{k+i}\right)\right]\\ &\quad+\sigma^2\left[\sum_{k\le i,j\le k+K-1}\Lambda_i^T\mathbf{Q}\Lambda_j+\sum_{k\le i<j\le k+K-1}\left(\mathbf{Q}\Lambda_i\Lambda_j+\Lambda_j^T\Lambda_i^T\mathbf{Q}\right)+\cdots\right].\end{aligned}$$

Note that $\|\Lambda_i\|\le\|\mathbf{LH}\|$ and the possible combinations are finite due to the boundedness of K; thus, there exists a constant $c_2>0$ such that the last term on the right-hand side of the last inequality over σ^2 is bounded by c_2I. Moreover, $\mathbf{Q}$ is a positive definite matrix; thus, there is a suitable constant $c_3>0$ such that $c_3\mathbf{Q}\le I$. Then, we have

$$\Phi^T_{k+K-1,k}\mathbf{Q}\Phi_{k+K-1,k}\le\mathbf{Q}-\sigma c_1I+\sigma^2c_2I\le\mathbf{Q}-\left[\sigma c_1-\sigma^2c_2\right]c_3\mathbf{Q}$$

as long as σ is small enough such that $\sigma c_1-\sigma^2c_2>0$ and $\sigma c_1c_3<1$. In such a case, denote $\eta=1-[\sigma c_1-\sigma^2c_2]c_3$, and it is clear that

$$\Phi^T_{k+K-1,k}\mathbf{Q}\Phi_{k+K-1,k}\le\eta\mathbf{Q},\quad 0<\eta<1,\quad\forall k. \tag{32}$$

The proof is completed. □

Remark 12. It is worth pointing out that the proof of Lemma 3 is quite technical; however, the inherent principle is not so complicated. Specifically, the introduction of the positive definite matrix $\mathbf{Q}$ is to make well-defined expressions in the analysis. The contract effect of $\Phi_{k+K-1,k}$ can be interpreted as follows. Λ_k is a block lower triangular matrix, and then, $I-\sigma\Lambda_k$ is a block lower triangular matrix with its eigenvalues being $1-\sigma\gamma_k(t+\tau)\lambda_{t,i}$, $1\le i\le p$, $0\le t\le N-\tau$, where $\lambda_{t,i}$ denotes the eigenvalue of $L_tC_{t+\tau}A^{t+\tau-1}_{t+1}B_t$. Therefore, when $\gamma_k(t+\tau)=0$, the corresponding eigenvalue of $I-\sigma\Lambda_k$ is 1, implying that no contraction occurs but neither any expansion occurs; when $\gamma_k(t+\tau)=1$, the corresponding eigenvalue of $I-\sigma\Lambda_k$ is less than 1 provided that the eigenvalues $\lambda_{t,i}$ are positive and σ is small enough, implying a contraction. The bounded-length assumption on successive data dropouts actually guarantees the infinitely often contractions along the iteration axis.

Remark 13. From the technical viewpoint, the parameter σ can be solved from the relationship $\sigma c_1-\sigma^2c_2>0$ and $\sigma c_1c_3<1$, ie, $\sigma<\min\{c_1c_2^{-1},(c_1c_3)^{-1}\}$. Apparently, σ should be small enough when little information of the system is known. However, a small σ value would render a large value of η, which limits the contraction effect. Thus, there is a trade-off in selecting parameter σ. In addition, the proof provides a rather conservative estimation of η while the actual contract influence is usually more efficient.

With the help of Lemma 3, we can give the convergence for the input sequence now.

Theorem 1. *Consider the noise-free linear system (18) and the ILC update law (19), where the random data dropouts follow the RSM case. Assume Assumption 1 holds. Then, the input sequence* $\{u_k(t)\}$, $t=0,\cdots,N-\tau$, *achieves both mean square convergence and almost sure convergence to the desired input* $u_d(t)$, $t=0,\ \dots\ ,N-\tau$, *if the learning gain matrix* L_t *satisfies that* $-L_tC_{t+\tau}A^{t+\tau-1}_{t+1}B_t$ *is a Hurwitz matrix and* σ *is small enough.*

Proof. The proof is carried out based on the inherent convergence principle that there exists at least one contraction during any K successive iterations. To this end, we can group the iteration number by a modulo operator with respect to K; that is, all iterations are divided into K subsets, $\{iK+j,i\ge0\}$, $0\le j\le K-1$. Then, we show the strict contraction mapping of the input sequence with the subscripts valued in each subset given above.

Define a weighted norm of δU_k as $V_k=\|\delta U_k\|_{\mathbf{Q}}\triangleq\left(\delta U_k\right)^T\mathbf{Q}\delta U_k$, which can be regarded as a Lyapunov function. Then, $\forall 0\le j\le K-1$, we have

$$\begin{aligned}V_{iK+j}&=\left(\delta U_{iK+j}\right)^T\mathbf{Q}\delta U_{iK+j}\\&=\left(\Phi_{iK+j-1,(i-1)K+j}\delta U_{(i-1)K+j}\right)^T\mathbf{Q}\Phi_{iK+j-1,(i-1)K+j}\delta U_{(i-1)K+j}\\&=\left(\delta U_{(i-1)K+j}\right)^T\Phi^T_{iK+j-1,(i-1)K+j}\mathbf{Q}\Phi_{iK+j-1,(i-1)K+j}\delta U_{(i-1)K+j}\\&\le\eta\left(\delta U_{(i-1)K+j}\right)^T\mathbf{Q}\delta U_{(i-1)K+j}\\&=\eta V_{(i-1)K+j},\quad i\ge1,\end{aligned}$$

where Equation 27 and Lemma 3 are used. Consequently, we have

$$\mathbb{E}\|\delta U_{iK+j}\|_{\mathbf{Q}} \leq \eta \mathbb{E}\|\delta U_{(i-1)K+j}\|_{\mathbf{Q}}, \quad \forall 0 \leq j \leq K-1, \quad i \geq 1. \tag{33}$$

Then, it directly leads to that

$$\mathbb{E}\|\delta U_{iK+j}\|_{\mathbf{Q}} \leq \eta^i \mathbb{E}\|\delta U_j\|_{\mathbf{Q}}, \quad 0 \leq j \leq K-1. \tag{34}$$

Meanwhile, following the same idea in Lemma 3, the weighted norms of the inputs for the first K iterations, ie, δU_j with $0 \leq j \leq K-1$, are bounded by the initial one. That is, $\forall 0 \leq j \leq K-1$, we have

$$\begin{aligned} V_j &= (\delta U_j)^T \mathbf{Q} \delta U_j \\ &= (\delta U_0)^T (I - \sigma \Lambda_0^T) \cdots (I - \sigma \Lambda_{j-1}^T) \mathbf{Q} (I - \sigma \Lambda_{j-1}) \cdots (I - \sigma \Lambda_0) \delta U_0 \\ &\leq (\delta U_0)^T \mathbf{Q} \delta U_0 = \|\delta U_0\|_{\mathbf{Q}}, \end{aligned}$$

where U_0 denotes the initial input. Incorporating with Equation 34, we evidently derive

$$\mathbb{E}\|\delta U_{iK+j}\|_{\mathbf{Q}} \underset{i \to \infty}{\longrightarrow} 0, \quad 0 \leq j \leq K-1. \tag{35}$$

Note that $\mathbf{Q}$ is a fixed positive definite matrix; therefore, a direct corollary of Equation 35 is that $\lim_{k\to\infty} \mathbb{E}\|\delta U_k\|^2 = 0$. In other words, the mean square convergence of the update law is established.

Next, we move to show the almost sure convergence. Recalling inequality (34) and noting that $\mathbf{Q}$ is a positive definite matrix, we have

$$\mathbb{E}\|\delta U_{iK+j}\|^2 \leq \lambda_{\min}^{-1}(\mathbf{Q}) \eta^i \mathbb{E}\|\delta U_j\|_{\mathbf{Q}}, \quad 0 \leq j \leq K-1, \tag{36}$$

where $\lambda_{\min}(\cdot)$ denotes the smallest eigenvalue of its indicated matrix. It follows that

$$\begin{aligned} \sum_{i=0}^{\infty} \mathbb{E}\|\delta U_{iK+j}\| &\leq \sum_{i=0}^{\infty} \lambda_{\min}^{-1/2}(\mathbf{Q}) \eta^{i/2} \left(\mathbb{E}\|\delta U_j\|_{\mathbf{Q}}\right)^{1/2} \\ &\leq \lambda_{\min}^{-1/2}(\mathbf{Q}) \left(\mathbb{E}\|\delta U_0\|_{\mathbf{Q}}\right)^{1/2} \sum_{i=0}^{\infty} \eta^{i/2} \\ &= \lambda_{\min}^{-1/2}(\mathbf{Q}) \left(\mathbb{E}\|\delta U_0\|_{\mathbf{Q}}\right)^{1/2} \frac{1}{1-\eta^{1/2}} < \infty, \end{aligned}$$

which further yields

$$\sum_{k=0}^{\infty} \mathbb{E}\|\delta U_k\| = \sum_{j=0}^{K-1} \sum_{i=0}^{\infty} \mathbb{E}\|\delta U_{iK+j}\| < \infty.$$

Then, by the Markov inequality, for any $\epsilon > 0$, we have

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\|\delta U_k\| > \epsilon\right) \leq \sum_{k=1}^{\infty} \frac{\mathbb{E}\|\delta U_k\|}{\epsilon} < \infty.$$

This fact leads to $\mathbb{P}\left(\|\delta U_k\| > \epsilon, \text{i.o.}\right) = 0$ by the Borel-Cantelli lemma, $\forall \epsilon > 0$, where "i.o." is short for "infinitely often." That is, $\mathbb{P}\left(\lim_{k\to\infty} \|\delta U_k\| = 0\right) = 1$. In other words, δU_k converges to zero almost surely. This completes the proof. □

Remark 14. In this section, the noise-free system is taken into account; therefore, the precise convergence of the input sequence ensures that the system output $y_k(t)$ can precisely track the desired reference $y_d(t)$, $\forall t$, with the help of Assumption 1. Moreover, it is noticed from Equation 34 that the update law (19) for the noise-free system ensures an exponential convergence speed. Meanwhile, such exponential convergence speed enables us to establish the almost sure convergence based on the Borel-Cantelli lemma.

4.2 | BVM case

When considering the BVM case, the technical lemma for the RSM case in the last subsection, ie, Lemma 3, is no longer valid due to the inherent randomness of data dropouts. However, in such case, the statistical property of the random variable $\gamma_k(t)$ is valuable for establishing the convergence results. Moreover, in the BVM assumption, the data dropout variable $\gamma_k(t)$ is independent along the iteration axis, that is, for different iteration numbers $k \neq l$, $\gamma_k(t)$ is independent of $\gamma_l(t)$, $\forall t$. Such independence will be used in the convergence analysis as follows.

Theorem 2. *Consider the noise-free linear system (18) and the ILC update law (19), where the random data dropouts follow the BVM case. Assume Assumption 1 holds. Then, the input sequence* $\{u_k(t)\}$, $t = 0, \ldots, N-\tau$, *achieves both mean square convergence and almost sure convergence to the desired input* $u_d(t)$, $t = 0, \ldots, N-\tau$, *if the learning gain matrix* L_t *satisfies that* $-L_tC_{t+\tau}A_{t+1}^{t+\tau-1}B_t$ *is a Hurwitz matrix and* σ *is small enough.*

Proof. We still apply the weighted norm of δU_k, $V_k = \|\delta U_k\|_{\mathbf{Q}} = (\delta U_k)^T\mathbf{Q}\delta U_k$. Then, we have

$$\begin{aligned} V_{k+1} &= \left(\delta U_{k+1}\right)^T\mathbf{Q}\delta U_{k+1} \\ &= \left(\delta U_k\right)^T(I-\sigma\Lambda_k)^T\mathbf{Q}(I-\sigma\Lambda_k)\delta U_k. \end{aligned} \tag{37}$$

In the BVM case, the data dropout is independent along the iteration axis, while δU_k is constructed based on the information of the $(k-1)$th iteration; thus, $I-\sigma\Lambda_k$ is independent of δU_k in Equation 37. Consequently, taking the mathematical expectation to both sides of Equation 37 leads to that

$$\begin{aligned} \mathbb{E}\|\delta U_{k+1}\|_{\mathbf{Q}} &= \mathbb{E}\left[\left(\delta U_k\right)^T(I-\sigma\Lambda_k)^T\mathbf{Q}(I-\sigma\Lambda_k)\delta U_k\right] \\ &= \mathbb{E}\left[\left(\delta U_k\right)^T\mathbb{E}\Big((I-\sigma\Lambda_k)^T\mathbf{Q}(I-\sigma\Lambda_k)\Big)\delta U_k\right]. \end{aligned} \tag{38}$$

Notice that

$$\begin{aligned} \mathbb{E}\Big((I-\sigma\Lambda_k)^T\mathbf{Q}(I-\sigma\Lambda_k)\Big) &= \mathbb{E}\Big(\mathbf{Q}-\sigma\left(\Lambda_k^T\mathbf{Q}+\mathbf{Q}\Lambda_k^T\right)+\sigma^2\Lambda_k^T\mathbf{Q}\Lambda_k^T\Big) \\ &= \mathbf{Q}-\sigma\mathbb{E}\left(\Lambda_k^T\mathbf{Q}+\mathbf{Q}\Lambda_k^T\right)+\sigma^2\mathbb{E}\Lambda_k^T\mathbf{Q}\Lambda_k^T. \end{aligned} \tag{39}$$

Recalling the definition of Λ_k^T, it is evident that $\mathbb{E}\Lambda_k = \bar{\gamma}\mathbf{LH}$. Incorporating with Equation 25 leads to that

$$\mathbb{E}\left(\Lambda_k^T\mathbf{Q}+\mathbf{Q}\Lambda_k^T\right) = \bar{\gamma}I. \tag{40}$$

On the other hand, there exists a suitable constant $c_4 > 0$ such that

$$\mathbb{E}\Lambda_k^T\mathbf{Q}\Lambda_k^T = \sum_{i=1}^{\kappa}\mathbb{P}\left(\Lambda_k = \Lambda^{(i)}\right)(\Lambda^{(i)})^T\mathbf{Q}\Lambda^{(i)} \leq c_4 I, \tag{41}$$

where $\mathbb{P}\left(\Lambda_k = \Lambda^{(i)}\right)$ denotes the probability that Λ_k is valued to be $\Lambda^{(i)}$ and $\sum_{i=1}^{\kappa}\mathbb{P}\left(\Lambda_k = \Lambda^{(i)}\right) = 1$.

From Equations 39, 40, and 41, it follows that

$$\mathbb{E}\Big((I-\sigma\Lambda_k)^T\mathbf{Q}(I-\sigma\Lambda_k)\Big) \leq \mathbf{Q}-\sigma(\bar{\gamma}-\sigma c_4)I. \tag{42}$$

Using the fact that $c_3\mathbf{Q} \leq I$ given in the last subsection, where $c_3 > 0$ is a suitable constant, and substituting Equation 42 into Equation 38, we have

$$\mathbb{E}\|\delta U_{k+1}\|_{\mathbf{Q}} \leq \left(1-\sigma(\bar{\gamma}-\sigma c_4)c_3\right)\mathbb{E}\|\delta U_k\|_{\mathbf{Q}}, \tag{43}$$

and consequently, we have a contraction mapping of $\mathbb{E}\|\delta U_k\|_{\mathbf{Q}}$ as

$$\mathbb{E}\|\delta U_{k+1}\|_{\mathbf{Q}} \leq \eta_1\mathbb{E}\|\delta U_k\|_{\mathbf{Q}}, \quad 0 < \eta_1 < 1,$$

where $\eta_1 \triangleq 1-\sigma(\bar{\gamma}-\sigma c_4)c_3$, as long as we select parameter σ to be small enough such that $\bar{\gamma}-\sigma c_4 > 0$ and $\sigma\bar{\gamma}c_3 < 1$.

Following similar steps to the proof of Theorem 1, we can obtain mean square convergence and almost sure convergence of zero for the input error δU_k. This completes the proof. □

Remark 15. The condition on parameter σ is given by 2 inequalities, ie, $\bar{\gamma}-\sigma c_4 > 0$ and $\sigma\bar{\gamma}c_3 < 1$, which leads to $\sigma < \bar{\gamma}c_4^{-1}$ and $\sigma < \bar{\gamma}^{-1}c_3^{-1}$. Since $\bar{\gamma} < 1$, the second range can be reduced to $\sigma < c_3^{-1}$. Thus, $\sigma < \min\{\bar{\gamma}c_4^{-1}, c_3^{-1}\}$. From this formulation, we find that the DDR, ie, the average level of data dropouts along the iteration axis, has an important influence on the selection of parameter σ. Roughly speaking, the smaller the DDR $\bar{\gamma}$, the smaller the parameter σ. Meanwhile, as we have previously explained, smaller selection of σ renders a slower convergence speed. This observation coincides with our intuitive recognition of the phenomenon that heavy data dropouts would lead to slower convergence of the ILC algorithms.

4.3 | MCM case

In this subsection, we move to consider the MCM case. The MCM case is more general than the BVM case as the independence property of $\gamma_k(t)$ along the iteration axis is no longer valid in the MCM case. Consequently, the separation of δU_k and Λ_k in Equation 38 is not applicable for the MCM case. This is the motivation of the convergence analysis proposed in this subsection. In fact, our objective in this subsection is to derive a similar contraction mapping for the MCM case.

With the same design condition of the learning gain matrices L_t given above, we have the following theorem for the MCM case.

Theorem 3. *Consider the noise-free linear system (18) and the ILC update law (19), where the random data dropouts follow the MCM case. Assume Assumption 1 holds. Then, the input sequence $\{u_k(t)\}$, $t = 0, \cdots, N-\tau$, achieves both mean square convergence and almost sure convergence to the desired input $u_d(t)$, $t = 0, \cdots, N-\tau$, if the learning gain matrix L_t satisfies that $-L_t C_{t+\tau} A_{t+1}^{t+\tau-1} B_t$ is a Hurwitz matrix and σ is small enough.*

Proof. Note that the matrix Λ_k is valued from the set $\mathfrak{S} = \{\Lambda^{(1)}, \cdots, \Lambda^{(\kappa)}\}$. We first point out that the evolution of Λ_k also forms a Markov chain.

In the MCM case, the random variable $\gamma_k(t)$ forms a Markov chain along the iteration axis, $\forall t$. From the definition of the Markov chain, we obtain $\mathbb{P}\big(\gamma_k(t) = r_k^t | \gamma_{k-1}(t) = r_{k-1}^t, \cdots, \gamma_1(t) = r_1^t\big) = \mathbb{P}\big(\gamma_k(t) = r_k^t | \gamma_{k-1}(t) = r_{k-1}^t\big)$, $r_k^t \in \{0, 1\}$, $\forall k, t$. Moreover, for different time instants $i \neq j$, $\gamma_k(i)$ is independent of $\gamma_k(j)$. Thus, we have

$$\begin{aligned}
&\mathbb{P}\big(\gamma_k(\tau) = r_k^\tau, \cdots, \gamma_k(N) = r_k^N | \gamma_{k-1}(\tau) = r_{k-1}^\tau, \cdots, \gamma_{k-1}(N) = r_{k-1}^N, \cdots, \gamma_1(\tau) = r_1^\tau, \cdots, \gamma_1(N) = r_1^N\big) \\
&= \mathbb{P}\big(\gamma_k(\tau) = r_k^\tau, \cdots, \gamma_k(N) = r_k^N | \gamma_{k-1}(\tau) = r_{k-1}^\tau, \cdots, \gamma_{k-1}(N) = r_{k-1}^N\big).
\end{aligned}$$

Then, the evolution of Γ_k along the iteration axis can also be characterized by a Markov chain and so is Λ_k. Denote the stationary transition probability matrix $(p_{ij})_{1\le i,j\le\kappa}$ with

$$p_{ij} = \mathbb{P}\big(\Lambda_k = \Lambda_k^{(j)} | \Lambda_{k-1} = \Lambda^{(i)}\big).$$

It is evident that $\min_{1\le i\le\kappa} p_{i1} > 0$.

Apply the weighted norm to δU_k, $V_k = \|\delta U_k\|_{\mathbf{Q}} = \delta U_k^T \mathbf{Q} \delta U_k$. Then, we have

$$\begin{aligned}
V_{k+1} &= \big(\delta U_{k+1}^T\big)^T \mathbf{Q} \delta U_{k+1} \\
&= \big(\delta U_k - \sigma\Lambda_k\delta U_k\big)^T \mathbf{Q}\big(\delta U_k - \sigma\Lambda_k\delta U_k\big) \\
&= \big(\delta U_k\big)^T \mathbf{Q}\delta U_k - \sigma\big(\delta U_k\big)^T\big[\Lambda_k^T\mathbf{Q} + \mathbf{Q}\Lambda_k\big]\delta U_k + \sigma^2\big(\delta U_k\big)^T\Lambda_k^T\mathbf{Q}\Lambda_k\delta U_k.
\end{aligned} \tag{44}$$

Note that δU_k is no longer independent of Λ_k. In order to make a separation, denote the σ-algebra

$$\mathcal{F}_k' = \sigma\big(x_j(0), u_j(i), y_j(i), \gamma_j(i), 1 \le j \le k-1, 1 \le i \le N\big)$$

(ie, the set of all events induced by these random variables) for $k \ge 1$. Taking the conditional expectation to both sides of Equation 44 with respect to $\mathcal{F}_k'$ leads to

$$\mathbb{E}\big(V_{k+1}|\mathcal{F}_k'\big) = V_k - \sigma\big(\delta U_k\big)^T\mathbb{E}\big(\Lambda_k^T\mathbf{Q} + \mathbf{Q}\Lambda_k|\mathcal{F}_k'\big)\delta U_k + \sigma^2\big(\delta U_k\big)^T\mathbb{E}\big(\Lambda_k^T\mathbf{Q}\Lambda_k|\mathcal{F}_k'\big)\delta U_k. \tag{45}$$

Recalling the stationary transition probability matrix $(p_{ij})_{1\le i,j\le\kappa}$, we have that

$$\mathbb{E}\big(\Lambda_k^T\mathbf{Q} + \mathbf{Q}\Lambda_k|\mathcal{F}_k'\big) \ge c_5 I, \tag{46}$$

where $c_5 > 0$ is a suitable constant. On the other hand, note that $\mathbf{Q}$ is a positive definite matrix; thus, there exists $c_6 > 0$ such that

$$\mathbb{E}\big(\Lambda_k^T\mathbf{Q}\Lambda_k|\mathcal{F}_k'\big) \le c_6 I. \tag{47}$$

Using the fact that $c_3\mathbf{Q} \le I$ and from Equations 45, 46, and 47, we have

$$\mathbb{E}\big(V_{k+1}|\mathcal{F}_k'\big) \le \big(1 - (\sigma c_5 - \sigma^2 c_6)c_3\big)V_k. \tag{48}$$

As a result, we can select σ to be small enough such that $0 < (\sigma c_5 - \sigma^2 c_6)c_3 < 1$. Then, taking the mathematical expectation to both sides of Equation 48 implies that

$$\mathbb{E}V_{k+1} \le \eta_2 \mathbb{E}V_k, \tag{49}$$

where $\eta_2 \triangleq 1 - (\sigma c_5 - \sigma^2 c_6)c_3$. This inequality further implies that $\lim_{k\to\infty}\mathbb{E}(\delta U_k)^T\mathbf{Q}\delta U_k = 0$. Again, $\mathbf{Q}$ is a specified positive definite matrix; thus, we have $\lim_{k\to\infty}\mathbb{E}(\delta U_k)^T\delta U_k = 0$. The mean square convergence is thus obtained.

Now, we move to show the almost sure convergence. In fact, from Equation 48, we have

$$\mathbb{E}\left(V_{k+1}|\mathcal{F}'_k\right) \leq (1-\eta_2)V_k \leq V_k, \tag{50}$$

which yields that $\{V_k, k \geq 1\}$ forms a supermartingale. Moreover, V_k is nonnegative for all $k \geq 1$. Then, by the martingale convergence theorem,[45] we have that V_k converges to a limit almost surely. When converging in both mean square sense and almost sure sense, the limitation should be identical. Therefore, V_k converges to zero almost surely, which further yields the convergence of δU_k to zero in the almost sure sense. This completes the proof. □

Remark 16. From the proofs of Theorems 1 to 3, we find that the convergence for the 3 cases follows the same inherent mechanism, that is, making a contraction mapping of a weighted norm of the input error vectors (ie, $\|\delta U_k\|_{\mathbf{Q}}$), which is also regarded as a Lyapunov function. The difference among the 3 cases lies in the iteration contraction length. Specifically, for the BVM and the MCM, contraction can be made for each iteration, whereas for the RSM, contraction can only be ensured jointly for K successive iterations, where K is defined in the RSM.

5 | CONVERGENCE OF THE STOCHASTIC LINEAR SYSTEM

In this section, we consider the stochastic linear system (1). We will give the mean square and almost sure convergence proofs for all 3 data dropout models. In addition, the update law is a slight variant of the classic P-type law (19), differing from the Kalman filtering–based algorithms.[31]

First, due to the existence of stochastic noise, the ILC update law (19) cannot guarantee stable convergence of the input sequence. Take the lifted form of Equation 20 for an intuitive understanding of this limitation. If the input sequence $\{U_k\}$ exists a stable convergence limitation, then taking the limitation to both sides of Equation 20 leads to $\lim_{k\to\infty} U_{k+1} = \lim_{k\to\infty} U_k + \lim_{k\to\infty} \sigma\Gamma_k \mathbf{L} E_k$. We can derive a simple corollary that $\lim_{k\to\infty} E_k = 0$. This corollary contradicts with the randomness of E_k in Equation 12. That is, the tracking error E_k consists of 2 parts, ie, $\mathbf{H}(U_d - U_k)$ and ξ_k; thus, it is impossible to derive $\lim_{k\to\infty} E_k = 0$.

Moreover, by Assumption 2, the stochastic noise cannot be predicted and eliminated by any algorithm; thus, we have to impose an additional mechanism to reduce the effect of noise along the iteration axis. As a matter of fact, it is well known that an appropriate decreasing gain for the correction term in updating processes is a necessary requirement to ensure convergence in the recursive computation for optimization, identification, and tracking of stochastic systems.[46,47] This fact is also illustrated in the ILC literature such as that by Saab,[31,35] in which the Kalman filtering–based method is proposed to deal with the stochastic systems, and the recursively computed learning gain matrix decreases to zero along the iteration axis. Inspired by this recognition, we replace the design parameter σ in Equation 19 with a decreasing sequence to cope with the stochastic noise. Specifically, the ILC update law for the stochastic system is modified as follows:

$$u_{k+1}(t) = u_k(t) + a_k \gamma_k(t+\tau) L_t e_k(t+\tau), \tag{51}$$

where the learning step-size $\{a_k\}$ is a decreasing sequence satisfying that

$$a_k \in (0,1), \quad a_k \to 0, \quad \sum_{k=1}^{\infty} a_k = \infty, \quad \sum_{k=1}^{\infty} a_k^2 < \infty, \quad \frac{1}{a_{k+1}} - \frac{1}{a_k} \to \chi > 0. \tag{52}$$

Remark 17. The decreasing step-size is an additional mechanism to cope with stochastic noise. Clearly, a basic selection $a_k = \alpha/k$ meets all the requirements of Equation 52 where $\alpha > 0$ is a tuning parameter. The inherent principle for introducing a_k is as follows. The tracking error E_k consists of 2 parts: the inaccurate tracking part $\mathbf{H}\delta U_k$ caused by the inaccurate input U_k and the stochastic noise part ξ_k. After sufficient learning iterations, it is believed that the inaccurate tracking part will significantly diminish such that the stochastic noise part dominates the tracking error. At this phase of the learning process, the decreasing step-size a_k will suppress stochastic noise to ensure stable convergence.

Remark 18. As has been shown by many results in stochastic control and optimization, the introduction of a decreasing step-size slows down the convergence speed. This fact is due to that the suppression effect of a_k is imposed not only on the stochastic noise but also on the correction information. In fact, it is a classic trade-off between the stable zero-error convergence and convergence speed for stochastic control. Roughly speaking, the exponential convergence speed for the noise-free case is no longer guaranteed. We can only ensure asymptotical convergence for stochastic systems. One may take interest in how to accelerate the convergence speed for practical applications. An acceleration approach with gain adaptation is given in the work of Shen and Xu,[48] which can be incorporated in the proposed algorithm. However, this is out of our scope; thus, we omit the details.

Similarly to the noise-free case, we lift the input along the time axis. The update law (51) is rewritten as follows:

$$U_{k+1} = U_k + a_k \Gamma_k \mathbf{L} E_k, \tag{53}$$

where Γ_k and $\mathbf{L}$ are given in Equations 21 and 22. Substracting both sides of Equation 53 from U_d, substituting the definition of $E_k = \mathbf{H}(U_d - U_k) - \xi_k$ (see Equation 12), and using the notation $\delta U_k = U_d - U_k$ lead to

$$\delta U_{k+1} = (I - a_k \Lambda_k)\delta U_k + a_k \xi_k, \tag{54}$$

where $\Lambda_k = \Gamma_k \mathbf{LH}$ is specified in the last section.

Before proceeding to the detailed convergence analysis for the 3 cases, we need to declare that the design condition for the learning gain matrix L_t remains the same as in the noise-free case. That is, the learning gain matrix L_t should satisfy that $-L_t C_{t+\tau} A_{t+1}^{t+\tau-1} B_t$ is a Hurwitz matrix.

In the following subsections, we will give the detailed convergence analyses of Equation 54 for the 3 models in turn. Similarly to the noise-free system case, the convergence is inherently guaranteed by the contraction property of $I - a_k \Lambda_k$. However, different from the noise-free system case, the sufficiently small constant σ is replaced by a decreasing gain sequence $\{a_k\}$. Consequently, the constant contraction for the noise-free system is no long valid for $I - a_k \Lambda_k$. Indeed, for the RSM, an elaborate estimation on the joint contraction effect is first given, whereas for the BVM and the MCM, such contraction effect is verified according to their probability properties. Then, we establish the asymptotical convergence based on the preliminary lemmas given in Section 3.4.

5.1 | RSM case

Similar to the noise-free case, we first give a decreasing property for the multiple products of $I - a_k \Lambda_k$ and then show the convergence with the help of such technical lemma.

Denote

$$\Psi_{m,n} = \left(I - a_m \Lambda_m\right) \cdots \left(I - a_n \Lambda_n\right), \quad m \geq n \tag{55}$$

and $\Psi_{m,m+1} \triangleq I$. Then, the estimate of $\Psi_{m,n}$ is given in the following lemma.

Lemma 4. *Consider the matrix product (55). If the learning gain matrix L_t satisfies that $-L_t C_{t+\tau} A_{t+1}^{t+\tau-1} B_t$ is a Hurwitz matrix, $\forall t$, then there exist constants $c_7 > 0$ and $c_8 > 0$ such that, for $m > n + K$, we have*

$$\|\Psi_{m,n}\| \leq c_7 \exp\left(-c_8 \sum_{i=n}^{m} a_i\right), \quad \forall n \geq 1. \tag{56}$$

Proof. First, we recall that $(\mathbf{LH})^T \mathbf{Q} + \mathbf{QLH} = I$ and $(\Lambda^{(i)})^T \mathbf{Q} + \mathbf{Q} \Lambda^{(i)} \geq 0$ for $i = 2, \ldots, \kappa$ (see Equations 25 and 26). The RSM assumption results in that

$$\left(\sum_{i=0}^{K-1} \Lambda_{k+i}\right)^T \mathbf{Q} + \mathbf{Q}\left(\sum_{i=0}^{K-1} \Lambda_{k+i}\right) \geq c_1 I, \quad \forall k \geq 0. \tag{57}$$

Moreover, from Equation 52, for $1 \leq i \leq K$, we have

$$\frac{a_{m-i}}{a_m} - 1 = a_{m-i}\left(\frac{1}{a_m} - \frac{1}{a_{m-i}}\right) = O(a_m). \tag{58}$$

For any $m \geq n + K - 1$, we have

$$\begin{aligned}
\Psi_{m,n}^T \mathbf{Q} \Psi_{m,n} &= \Psi_{m-1,n}^T \big(I - a_m \Lambda_m\big)^T \mathbf{Q} \big(I - a_m \Lambda_m\big) \Psi_{m-1,n} \\
&= \Psi_{m-K,n}^T \big(I - a_{m-K+1} \Lambda_{m-K+1}\big)^T \cdots \big(I - a_m \Lambda_m\big)^T \mathbf{Q} \times \big(- a_m \Lambda_m\big) \cdots \big(I - a_{m-K+1} \Lambda_{m-K+1}\big) \Psi_{m-K,n} \\
&= \Psi_{m-K,n}^T \left[\mathbf{Q} - \left(\sum_{i=m-K+1}^{m} a_i \Lambda_i^T \mathbf{Q} + \sum_{i=m-K+1}^{m} a_i \mathbf{Q} \Lambda_i \right) + o(a_m) \right] \Psi_{m-K,n} \\
&= \Psi_{m-K,n}^T \left\{ \mathbf{Q} - a_m \left[\left(\sum_{i=m-K+1}^{m} \Lambda_i^T \right) \mathbf{Q} + \mathbf{Q} \left(\sum_{i=m-K+1}^{m} \Lambda_i \right) \right] + o(a_m) \right\} \Psi_{m-K,n},
\end{aligned} \tag{59}$$

where equality (58) is invoked.

Noticing $0 < a_m < 1$ for large enough m and using Equation 57, we have

$$\begin{aligned}
\Psi_{m,n}^T \mathbf{Q} \Psi_{m,n} &\leq \Psi_{m-K,n}^T \big(\mathbf{Q} - a_m c_1 I + o(a_m)\big) \Psi_{m-K,n} \\
&\leq \Psi_{m-K,n}^T \mathbf{Q}^{\frac{1}{2}} \big(I - a_m c_1 \mathbf{Q}^{-1} + o(a_m)\big) \mathbf{Q}^{\frac{1}{2}} \Psi_{m-K,n} \\
&\leq \Psi_{m-K,n}^T \mathbf{Q}^{\frac{1}{2}} \left(I - \frac{c_1}{K} \mathbf{Q}^{-1} \sum_{i=m-K+1}^{m} a_i + o(a_m) \right) \mathbf{Q}^{\frac{1}{2}} \Psi_{m-K,n} \\
&\leq \left(1 - \frac{c_1}{K} \lambda_{\min}(\mathbf{Q}^{-1}) \sum_{i=m-K+1}^{m} a_i + o(a_m) \right) \Psi_{m-K,n}^T \mathbf{Q} \Psi_{m-K,n} \\
&\leq \exp\left(-c_9 \sum_{i=m-K+1}^{m} a_i \right) \Psi_{m-K,n}^T \mathbf{Q} \Psi_{m-K,n}
\end{aligned} \tag{60}$$

for sufficiently large n, where c_8 is a positive constant.

Therefore, for sufficiently large n, for example, for $n \geq n_0$ and $m \geq n + K$, we have

$$\Psi_{m,n}^T \mathbf{Q} \Psi_{m,n} \leq c_{10} \exp\left(-c_9 \sum_{i=n}^{m} a_i \right) I \quad \text{with} \quad c_{10} > 0, \tag{61}$$

which, by noticing the definition of $\mathbf{Q} > 0$, implies

$$\|\Psi_{m,n}\| \leq c_{11} \exp\left(-\frac{c_9}{2} \sum_{i=n}^{m} a_i \right) \quad \text{with} \quad c_{11} > 0. \tag{62}$$

Consequently, for $\forall n \geq n_0 + K$, $\forall n > 0$, by Equation 62 and the definition $\Psi_{m,m+1} \triangleq I$, we have

$$\|\Psi_{m,n}\| = \|\Psi_{m,n_0}\| \cdot \|\Psi_{n_0-1,n}\| \leq c_7 \exp\left(-c_8 \sum_{i=n}^{m} a_i \right), \tag{63}$$

where c_7 is a suitable constant, and $c_8 = c_9/2$. The proof is completed. □

Remark 19. Comparing the estimations of the corresponding product (30) for the noise-free case and Equation 56 for the noise case, we can have a clear understanding of the difference between the fixed step-size σ and the decreasing step-size a_k. Specifically, these 2 estimations are consistent as if we replace the decreasing step-size a_k with the fixed but small enough σ, estimation (56) actually turns into estimation (30). In other words, Equation 30 can be regarded as a special case of Equation 56.

Now, we can move to show the convergence for the RSM case.

Theorem 4. *Consider the stochastic linear system (1) and the ILC update law (51), where the random data dropouts follow the RSM case. Assume Assumptions 1 and 2 hold. Then, the input sequence* $\{u_k(t)\}$, $t = 0, \cdots, N - \tau$, *achieves both mean square convergence and almost sure convergence to the desired input* $u_d(t)$, $t = 0, \cdots, N - \tau$, *if the learning gain matrix* L_t *satisfies that* $-L_t C_{t+\tau} A_{t+1}^{t+\tau-1} B_t$ *is a Hurwitz matrix.*

Proof. The proof is carried out through grouping the iterations by a modulo operator with respect to K. To this end, all iterations are divided into K subsets, $\{iK + j, i \geq 0\}$, $0 \leq j \leq K - 1$. Now, we check the contraction for successive K iterations, that is, we check the convergence for each subset.

From Equation 54, it follows, $\forall 0 \le j \le K-1$, that

$$\delta U_{iK+j} = \Psi_{iK+j-1,(i-1)K+j}\delta U_{(i-1)K+j} + \sum_{l=0}^{K-1} \Psi_{iK+j-1,(i-1)K+j+l+1} a_{(i-1)K+j+l}\xi_{(i-1)K+j+l}. \tag{64}$$

Apply the weighted norm $V_k = \|\delta U_k\|_{\mathbf{Q}} = \delta U_k^T \mathbf{Q}\delta U_k$. We have that

$$\begin{aligned} V_{iK+j} &= \delta U_{iK+j}^T \mathbf{Q}\delta U_{iK+j} \\ &= \left(\Psi_{iK+j-1,(i-1)K+j}\delta U_{(i-1)K+j}\right)^T \mathbf{Q}\Psi_{iK+j-1,(i-1)K+j}\delta U_{(i-1)K+j} + 2\left(\Psi_{iK+j-1,(i-1)K+j}\delta U_{(i-1)K+j}\right)^T \mathbf{Q}\phi_* + \phi_*^T \mathbf{Q}\phi_*, \end{aligned} \tag{65}$$

where

$$\phi_* \triangleq \sum_{l=0}^{K-1} \Psi_{iK+j-1,(i-1)K+j+l+1} a_{(i-1)K+j+l}\xi_{(i-1)K+j+l}. \tag{66}$$

From the proof of Lemma 4, it follows that

$$\Psi_{iK+j-1,(i-1)K+j}^T \mathbf{Q}\Psi_{iK+j-1,(i-1)K+j} \le (1 - c_{12}a_{iK+j-1} + c_{13}a_{iK+j-1}^2)\mathbf{Q}, \tag{67}$$

which implies that

$$\left(\Psi_{iK+j-1,(i-1)K+j}\delta U_{(i-1)K+j}\right)^T \mathbf{Q}\Psi_{iK+j-1,(i-1)K+j}\delta U_{(i-1)K+j} \le (1 - c_{12}a_{iK+j-1} + c_{13}a_{iK+j-1}^2)\|\delta U_{(i-1)K+j}\|_{\mathbf{Q}}. \tag{68}$$

Noticing that ϕ_* is a sum of random noise and that the noise variables are independent of the data dropout variables, we have

$$\begin{aligned} \mathbb{E}\left(\Psi_{iK+j-1,(i-1)K+j}\delta U_{(i-1)K+j}\right)^T \mathbf{Q}\phi_* &= \left(\mathbb{E}\Psi_{iK+j-1,(i-1)K+j}\delta U_{(i-1)K+j}\right)^T \mathbf{Q}\left(\mathbb{E}\phi_*\right) \\ &= \left(\mathbb{E}\Psi_{iK+j-1,(i-1)K+j}\delta U_{(i-1)K+j}\right)^T \mathbf{Q}\mathbb{E}\left(\mathbb{E}(\phi_* | \mathcal{F}''_{(i-1)K+j-1})\right) = 0, \end{aligned} \tag{69}$$

where the σ-algebra $\mathcal{F}''_k$ is augmented from $\mathcal{F}_k$ as $\mathcal{F}''_k = \sigma(x_i(t), u_i(t), y_i(t), w_i(t), v_i(t), \gamma_i(t), 1 \le i \le k, 0 \le t \le N)$. Moreover, by Assumption 2, the stochastic noise variables are conditionally independent along the iteration axis; thus, it follows that

$$\begin{aligned} \mathbb{E}\phi_*^T \mathbf{Q}\phi_* &= \mathbb{E}\left(\sum_{l=0}^{K-1} \Psi_{iK+j-1,(i-1)K+j+l+1} a_{(i-1)K+j+l}\xi_{(i-1)K+j+l}\right)^T \times \mathbf{Q}\left(\sum_{l=0}^{K-1} \Psi_{iK+j-1,(i-1)K+j+l+1} a_{(i-1)K+j+l}\xi_{(i-1)K+j+l}\right) \\ &= \mathbb{E}\left(\sum_{l=0}^{K-1} a_{(i-1)K+j+l}^2 \xi_{(i-1)K+j+l}^T \Psi_{iK+j-1,(i-1)K+j+l+1}^T \times \mathbf{Q}\Psi_{iK+j-1,(i-1)K+j+l+1}\xi_{(i-1)K+j+l}\right) \\ &\le \sum_{l=0}^{K-1} a_{(i-1)K+j+l}^2 c_7^2 \exp\left(-2c_8 \sum_{i=(i-1)K+j+l+1}^{iK+j-1}\right)\mathbb{E}\|\xi_{(i-1)K+j+l}\|^2 \le a_{iK+j-1}^2 c_{14}, \end{aligned} \tag{70}$$

where c_{14} is a suitable constant such that $c_{14} \ge c_7^2 \sup_k \mathbb{E}\|\xi_k\|^2 \sum_{l=0}^{K-1}\left(a_{(i-1)K+j+l}^2 / a_{iK+j-1}^2\right)$.

Taking the mathematical expectation to both sides of Equation 65 and substituting Equations 67 to 70, we have

$$\mathbb{E}V_{iK+j} \le (1 - c_{12}a_{iK+j-1})\mathbb{E}V_{(i-1)K+j} + c_{13}a_{iK+j-1}^2(\mathbb{E}V_{(i-1)K+j} + c_{14}/c_{13}), \quad \forall\, 0 \le j \le K-1. \tag{71}$$

Comparing Equation 71 with Equation 15 in Lemma 1, it is found that $\mathbb{E}V_{iK+j}$, a_{iK+j-1} (with respect to recursive index i), c_{12}, c_{13}, and c_{14}/c_{13} correspond to ϑ_{k+1}, a_k (with respect to recursive index k), d_1, d_2, and d_3, respectively. Then, by Lemma 1, we have that $\lim_{i\to\infty}\mathbb{E}V_{iK+j} = 0$, $\forall 0 \le j \le K-1$. Moreover, incorporating with the fact that $\mathbf{Q}$ is a positive definite matrix, the mean square convergence is established for each subset of iteration number $\{iK+j, i \ge 0\}$, ie, $\lim_{i\to\infty}\mathbb{E}\|\delta U_{iK+j}\|^2 = 0$, $\forall 0 \le j \le K-1$. The mean square convergence of the input sequence $\{U_k, k \ge 1\}$ to the desired input U_d is thus obvious.

Next, we proceed to show the almost sure convergence of δU_k to zero. Taking a conditional expectation to Equation 65 with respect to σ-algebra $\mathcal{F}''_{(i-1)K+j-1}$, it follows that

$$\mathbb{E}\left(V_{iK+j} | \mathcal{F}''_{(i-1)K+j-1}\right) \le V_{(i-1)K+j} + c_{13}a_{iK+j-1}^2(V_{(i-1)K+j} + c_{14}/c_{13}), \quad \forall\, 0 \le j \le K-1. \tag{72}$$

Note that the 2 terms on the right-hand side of the last inequality, ie, $V_{(i-1)K+j}$ and $c_{13}a_{iK+j-1}^2(V_{(i-1)K+j} + c_{14}/c_{13})$, correspond to $X(n)$ and $Z(n)$ in Lemma 2, respectively. Moreover, it has been shown that $\mathbb{E}V_{(i-1)K+j}$ converges to zero as $i \to \infty$; thus, it is evident that

$$\sum_{i=0}^{\infty} \mathbb{E}\left[c_{13}a_{iK+j-1}^2(V_{(i-1)K+j} + c_{14}/c_{13})\right] \le \left(c_{13}\sup_i \mathbb{E}V_{(i-1)K+j} + c_{14}\right)\sum_{i=0}^{\infty} a_{iK+j-1}^2 < \infty. \tag{73}$$

In other words, the conditions in Lemma 2 are fulfilled. Therefore, it follows that V_{iK+j} converges almost surely as $i \to \infty$, $\forall j$. On the other hand, we have shown that δU_{iK+j} converges to zero in the mean square sense. Then, the almost surely convergent limitation of δU_{iK+j} should also be zero. The proof is completed. □

5.2 | BVM case

In this subsection, we give the convergence analysis for the BVM case. In such case, the deterministic contraction of the RSM is not valid; however, the independence of data dropouts would help establish the convergence similar to the last section.

Theorem 5. *Consider the stochastic linear system (1) and the ILC update law (51), where the random data dropouts follow the BVM case. Assume Assumptions 1 and 2 hold. Then, the input sequence $\{u_k(t)\}$, $t = 0, \cdots, N-\tau$, achieves both mean square convergence and almost sure convergence to the desired input $u_d(t)$, $t = 0, \cdots, N-\tau$, if the learning gain matrix L_t satisfies that $-L_t C_{t+\tau} A_{t+1}^{t+\tau-1} B_t$ is a Hurwitz matrix.*

Proof. Let us recall update law (53) as follows:

$$U_{k+1} = U_k + a_k \Gamma_k \mathbf{L} E_k.$$

Subtracting both sides of the last equation from U_d, we have

$$\begin{aligned}\delta U_{k+1} &= \delta U_k - a_k \Gamma_k \mathbf{L} E_k \\ &= \delta U_k - a_k \Gamma_k \mathbf{L}\mathbf{H} \delta U_k + a_k \Gamma_k \xi_k \\ &= \delta U_k - a_k \bar{\gamma} \mathbf{L}\mathbf{H} \delta U_k + a_k \left(\bar{\gamma} I - \Gamma_k\right) \mathbf{L}\mathbf{H} \delta U_k + a_k \Gamma_k \xi_k.\end{aligned} \tag{74}$$

Note that $\bar{\gamma} I$ is the mathematical expectation of Γ_k. Now, let us apply the weighted norm of δU_k, $V_k = \|\delta U_k\|_{\mathbf{Q}}$, ie,

$$\begin{aligned}V_{k+1} &= \delta U_{k+1}^T \mathbf{Q} \delta U_{k+1} \\ &= \delta U_k^T \mathbf{Q} \delta U_k + a_k^2 \bar{\gamma}^2 \delta U_k^T \left(\mathbf{L}\mathbf{H}\right)^T \mathbf{Q}\mathbf{L}\mathbf{H} \delta U_k + a_k^2 \xi_k^T \Gamma_k^T \mathbf{Q} \Gamma_k \xi_k + a_k^2 \delta U_k^T (\mathbf{L}\mathbf{H})^T \left(\bar{\gamma} I - \Gamma_k\right) \mathbf{Q} \left(\bar{\gamma} I - \Gamma_k\right) \left(\mathbf{L}\mathbf{H}\right) \delta U_k \\ &\quad - a_k \bar{\gamma} \delta U_k^T \left[\left(\mathbf{L}\mathbf{H}\right)^T \mathbf{Q} + \mathbf{Q}\mathbf{L}\mathbf{H}\right] \delta U_k + 2 a_k \left(\delta U_k - a_k \bar{\gamma} \mathbf{L}\mathbf{H} \delta U_k\right)^T \mathbf{Q} \left(\bar{\gamma} I - \Gamma_k\right) \left(\mathbf{L}\mathbf{H}\right) \delta U_k \\ &\quad + 2 a_k \left(\delta U_k - a_k \bar{\gamma} \mathbf{L}\mathbf{H} \delta U_k\right)^T \mathbf{Q} \Gamma_k \xi_k + 2 a_k^2 \delta U_k^T \left(\mathbf{L}\mathbf{H}\right)^T \left(\bar{\gamma} I - \Gamma_k\right) \mathbf{Q} \Gamma_k \xi_k.\end{aligned} \tag{75}$$

Note that U_k is constructed on the basis of the data from the $(k-1)$th iteration; thus, it is independent of the data dropout variable at the kth iteration, ie, Γ_k. This fact gives that

$$\mathbb{E}\left[\left(\delta U_k - a_k \bar{\gamma} \mathbf{L}\mathbf{H} \delta U_k\right)^T \mathbf{Q} \left(\bar{\gamma} I - \Gamma_k\right) \left(\mathbf{L}\mathbf{H}\right) \delta U_k\right] = 0. \tag{76}$$

Similarly, the independence of U_k, Γ_k, and ξ_k yields

$$\mathbb{E}\left[\left(\delta U_k - a_k \bar{\gamma} \mathbf{L}\mathbf{H} \delta U_k\right)^T \mathbf{Q} \Gamma_k \xi_k\right] = 0, \tag{77}$$

$$\mathbb{E}\left[\delta U_k^T \left(\mathbf{L}\mathbf{H}\right)^T \left(\bar{\gamma} I - \Gamma_k\right) \mathbf{Q} \Gamma_k \xi_k\right] = 0, \tag{78}$$

where Assumption 2 is applied.

Taking the mathematical expectation to both sides of Equation 75 and substituting Equations 76 to 78 as well as the Lyapunov equation $(\mathbf{L}\mathbf{H})^T \mathbf{Q} + \mathbf{Q}\mathbf{L}\mathbf{H} = I$, we have

$$\begin{aligned}\mathbb{E} V_{k+1} &= \mathbb{E} V_k - a_k \bar{\gamma} \mathbb{E} \delta U_k^T \delta U_k + a_k^2 \bar{\gamma}^2 \mathbb{E}\left[\delta U_k^T \left(\mathbf{L}\mathbf{H}\right)^T \mathbf{Q}\mathbf{L}\mathbf{H} \delta U_k\right] \\ &\quad + a_k^2 \mathbb{E}\left[\xi_k^T \Gamma_k^T \mathbf{Q} \Gamma_k \xi_k\right] + a_k^2 \mathbb{E}\left[\delta U_k^T (\mathbf{L}\mathbf{H})^T \left(\bar{\gamma} I - \Gamma_k\right) \mathbf{Q} \left(\bar{\gamma} I - \Gamma_k\right) \left(\mathbf{L}\mathbf{H}\right) \delta U_k\right].\end{aligned} \tag{79}$$

According to Assumption 2, there exists a suitable constant $c_{15} > 0$ such that

$$\mathbb{E}\left[\xi_k^T \Gamma_k^T \mathbf{Q} \Gamma_k \xi_k\right] < c_{15}. \tag{80}$$

Moreover, due to the positive definite property of $\mathbf{Q}$, there are $c_{16} > 0$ and $c_{17} > 0$ such that

$$\mathbb{E}\left[\delta U_k^T \left(\mathbf{L}\mathbf{H}\right)^T \mathbf{Q}\mathbf{L}\mathbf{H} \delta U_k\right] \le c_{16} \mathbb{E} V_k, \tag{81}$$

$$\mathbb{E}\left[\delta U_k^T (\mathbf{L}\mathbf{H})^T \left(\bar{\gamma} I - \Gamma_k\right) \mathbf{Q} \left(\bar{\gamma} I - \Gamma_k\right) \left(\mathbf{L}\mathbf{H}\right) \delta U_k\right] \le c_{17} \mathbb{E} V_k. \tag{82}$$

Substituting Equations 80 to 82 and the inequality $c_3\mathbf{Q} \leq I$ into Equation 79 leads to

$$\mathbb{E}V_{k+1} \leq (1 - a_k\bar{\gamma})\,\mathbb{E}V_k + a_k^2\left(c_{15} + (\bar{\gamma}^2 c_{16} + c_{17})\mathbb{E}V_k\right). \tag{83}$$

Then, it is evident that $\mathbb{E}V_k$ corresponds to ϑ_k in Lemma 1. Applying Lemma 1, it follows that $\lim_{k\to\infty}\mathbb{E}V_k = 0$, which further implies that $\mathbb{E}\|\delta U_k\|^2 = 0$ by the fact that $\mathbf{Q}$ is a positive definite matrix. The mean square convergence is thus obtained.

Next, we proceed to show the almost sure convergence with the help of Lemma 2. To this end, taking the conditional expectation to both sides of Equation 75 with respect to $\mathcal{F}''_{k-1}$, it follows that

$$\mathbb{E}\left(V_{k+1}|\mathcal{F}''_{k-1}\right) \leq V_k + a_k^2\left(c_{15} + (\bar{\gamma}^2 c_{16} + c_{17})V_k\right). \tag{84}$$

Condition (17) is easy to verify for the last term of the above inequality with the help of the mean square convergence. Therefore, by using Lemma 2, it gives that δU_k converges almost surely. Similar to the steps in the proof for Theorem 4, the almost sure convergence of the input sequence $\{U_k\}$ is verified. This completes the proof. □

5.3 | MCM case

In this subsection, the convergence analysis for the MCM case is given. As previously explained, the inherent difference between the BVM case and the MCM case is the iteration dependence of the data dropout in the MCM case. Thus, the proof can be carried out by modifying the step of taking the mathematical expectation in the proof of Theorem 5 as a conditional expectation. In the following, we will only give the main sketch of the proof, to save space.

Theorem 6. *Consider the stochastic linear system (1) and the ILC update law (51), where the random data dropouts follow the MCM case. Assume Assumptions 1 and 2 hold. Then, the input sequence* $\{u_k(t)\}$, $t = 0, \cdots, N-\tau$, *achieves both mean square convergence and almost sure convergence to the desired input* $u_d(t)$, $t = 0, \cdots, N-\tau$, *if the learning gain matrix* L_t *satisfies that* $-L_t C_{t+\tau} A_{t+1}^{t+\tau-1} B_t$ *is a Hurwitz matrix.*

Proof. Note that the data dropout is not independent along the iteration axis; thus, it is unsuitable to derive an expectation of Γ_k as we have done in Equation 74. In fact, the expression for δU_{k+1} is

$$\delta U_{k+1} = \delta U_k - a_k\Gamma_k\mathbf{LH}\delta U_k + a_k\Gamma_k\xi_k, \tag{85}$$

and then, the expansion of V_{k+1} is formulated as

$$\begin{aligned} V_{k+1} = {} & V_k + a_k^2\delta U_k^T\left(\mathbf{LH}\right)^T\Gamma_k^T\mathbf{Q}\Gamma_k\mathbf{LH}\delta U_k + a_k^2\xi_k^T\Gamma_k^T\Gamma_k\xi_k - a_k\delta U_k^T\left[\Lambda_k^T\mathbf{Q} + \mathbf{Q}\Lambda_k\right]\delta U_k \\ & + a_k\left[\delta U_k^T\mathbf{Q}\Gamma_k\xi_k + \xi_k^T\Gamma_k^T\mathbf{Q}\delta U_k\right] - a_k^2\left[\delta U_k^T\Lambda_k^T\mathbf{Q}\Gamma_k\xi_k + \xi_k^T\Gamma_k^T\mathbf{Q}\Lambda_k\delta U_k\right], \end{aligned} \tag{86}$$

where $\Lambda_k = \Gamma_k\mathbf{LH}$ has been previously defined.

Note that ξ_k is independent of other signals; thus, we have

$$\mathbb{E}\left(\delta U_k^T\mathbf{Q}\Gamma_k\xi_k + \xi_k^T\Gamma_k^T\mathbf{Q}\delta U_k|\mathcal{F}''_{k-1}\right) = 0, \tag{87}$$

$$\mathbb{E}\left(\delta U_k^T\Lambda_k^T\mathbf{Q}\Gamma_k\xi_k + \xi_k^T\Gamma_k^T\mathbf{Q}\Lambda_k\delta U_k|\mathcal{F}''_{k-1}\right) = 0. \tag{88}$$

Moreover, Γ_k is a bounded matrix; thus, there exists $c_{18} > 0$ such that

$$\mathbb{E}\left(\xi_k^T\Gamma_k^T\Gamma_k\xi_k|\mathcal{F}''_{k-1}\right) < c_{18}. \tag{89}$$

Furthermore, δU_k is adapted to $\mathcal{F}''_{k-1}$ according to the definition of update law, whereas the probability transition matrix for the stochastic matrix Λ_k has a positive probability of returning to $\Lambda^{(1)}$, ie, $\min_{1\leq i\leq\kappa}\mathbb{P}(\Lambda_k = \Lambda^{(1)}|\Lambda_{k-1} = \Lambda^{(i)}) > 0$ (see Section 4.3); therefore, there exists a constant $c_{19} > 0$ such that

$$\mathbb{E}\left(\Lambda_k^T\mathbf{Q} + \mathbf{Q}\Lambda_k|\mathcal{F}''_{k-1}\right) \geq c_{19}I. \tag{90}$$

Using Equations 87 to 90, we are able to derive from Equation 86 that

$$\mathbb{E}\left(V_{k+1}|\mathcal{F}''_{k-1}\right) \leq V_k - a_k c_{19}c_3V_k + a_k^2c_{20}V_k + a_k^2c_{18}, \tag{91}$$

where $c_{20} > 0$ is a constant such that

$$\mathbb{E}\left(\delta U_k^T\left(\mathbf{LH}\right)^T\Gamma_k^T\mathbf{Q}\Gamma_k\mathbf{LH}\delta U_k|\mathcal{F}''_{k-1}\right) \leq c_{20}V_k.$$

Then, taking the mathematical expectation to Equation 91, we have

$$\mathbb{E}V_{k+1} \leq (1 - a_k c_{19} c_3)\mathbb{E}V_k + a_k^2(c_{18} + c_{20}\mathbb{E}V_k). \tag{92}$$

From now on, the steps are similar to the proof of Theorem 5. The mean square and almost sure convergence of the input sequence $\{U_k\}$ to the desired input U_d can be obtained with the help of Lemmas 1 and 2. The proof is thus completed. □

5.4 | Further remarks

Remark 20. From the proofs in this section and in the last section, we may find the technical connections and differences among the 3 different cases. Specifically, the proof for the BVM case forms a basic procedure of the technical convergence analysis, in which, by taking the expectation of the random variable of the data dropouts, we can derive a separation formula of the input error δU_{k+1}, namely, the contraction mapping of the input error and 2 additional zero-expectation errors (see Equation 74 for details). Then, the convergence proof can be established with the help of Lemmas 1 and 2. That is, the additive formulation (74) plays a basic role for the following analysis. For the RSM case and the MCM case, this basic formula should be modified accordingly. In particular, for the RSM case, the 1-step/iteration contraction relationship in the BVM case (ie, $(I - a_k\bar{\gamma}\mathbf{LH})\delta U_k$ in Equation 74) has to be extended to K-step/iteration contraction relationship (see Equation 64 for a clear expression). This is also why we have to give a technical lemma for estimating $\Psi_{m,n}$ (see Lemma 4) before stating the main theorem. For the MCM case, the specific separation of Equation 74 cannot be derived due to the iteration dependence of data dropouts, but a similar contraction relationship can be obtained by taking a conditional expectation. In fact, this is the inherent difference between the BVM case and the MCM case, which originates from the model differences of the BVM and the MCM. To sum up, the RSM case and the MCM case are extensions of the BVM case from different aspects, and additional treatments are developed to complete the convergence analysis.

Remark 21. The essential step in the above proofs is to establish a decreasing trend of the weighted norm of the input error, ie, $\mathbb{E}V_k = \mathbb{E}\|\delta U_k\|_{\mathbf{Q}}$. For the noise-free system, it is seen that a monotonic decreasing trend for $\mathbb{E}V_k$ is derived, and the exponential convergence speed is thus guaranteed. Then, the almost sure convergence can be derived by applying the Borel-Cantelli lemma. For the noised system or the stochastic system, due to the existence of stochastic noise, it is impossible to reach the monotonic decreasing trend for $\mathbb{E}V_k$. However, a weak version of the decreasing trend can be established, ie, $\mathbb{E}V_{k+1} \leq (1 - a_k d_1)\mathbb{E}V_k + a_k^2 d_2(\mathbb{E}V_k + d_3)$. This formula implies that the main trend of $\mathbb{E}V_k$ is still decreasing, as shown by the term $(1 - a_k d_1)\mathbb{E}V_k$, but it could be involved with a faster attenuation term, as shown by the term $a_k^2 d_2(\mathbb{E}V_k + d_3)$. In such case, we can still ensure the convergence to zero of $\mathbb{E}V_k$. Moreover, the almost sure convergence is established with a weak version of the convergence theorem for a nonnegative supermartingale sequence. Specifically, let us revisit Lemma 2. If $Z(n)$ is removed from Equation 16, then $X(n)$ (corresponding to V_k in the following theorems) forms a supermartingale, and the almost sure convergence is thus guaranteed. Lemma 2 implies that the almost sure convergence is still true for $X(n)$ as long as the infinite sum of the additional term $Z(n)$ is finite in the mathematical expectation sense. To sum up, the mean square convergence and the almost sure convergence for the 3 models are established in a framework based on 2 technical lemmas.

Remark 22. It should be pointed out that although we do not provide a similar estimate of $\Psi_{m,n}$ for the BVM and MCM cases, the estimate in Lemma 4 is also valid for the latter 2 cases with $m \geq n$. Such derivations can be made following similar steps to the proof of Lemma 4. In fact, the similar conclusions have been merged into the convergence proofs for the latter cases (see the steps for deriving Equations 79 and 91). In fact, it is the decreasing property of $\Psi_{m,n}$ that essentially guarantees the convergence of the algorithms. In addition, the estimation of $\Psi_{m,n}$ also implies the convergence speed of the stochastic system case. Specifically, we may write that $\|\Psi_{m,1}\| = O\left(\exp\left(-\beta \sum_{k=1}^{m} a_k\right)\right)$. Let us select an alternative for $\{a_k\}$ as an illustration, ie, $a_k = 1/k$. It is well known that the mth harmonic number has an estimate $\sum_{k=1}^{m} a_k = O\left(\log m\right)$. Then, we have $\|\Psi_{m,1}\| = O\left(\exp\left(-\beta' \log m\right)\right) = O\left(m^{-\beta'}\right)$. This rate of convergence coincides with the basic knowledge of stochastic control.

Remark 23. From the design condition of the learning gain matrix, it is seen that the critical components for ensuring the convergence are the diagonal blocks of $\mathbf{LH}$, ie, $L_t C_{t+\tau} A_{t+1}^{t+\tau-1} B_t$, denoting the input/output coupling matrix, whereas the nondiagonal blocks of $\mathbf{LH}$ have little influence on the essential convergence. From this viewpoint, the results of this paper can be extended to affine nonlinear systems without significant efforts. The extension from linear systems to affine nonlinear systems has been reported in many existing papers. Here, we omit the tedious discussions to avoid repetition.

6 | ILLUSTRATIVE EXAMPLES

The main objective of this paper is to propose a general framework for the convergence analysis of ILC algorithms under various kinds of data dropout models. In the last 2 sections, the detailed analysis steps and techniques are elaborated. In this section, we verify the theoretical results with a time-varying linear system (A_t, B_t, C_t) where

$$A_t = \begin{bmatrix} 0.2\exp(-t/100) & -0.6 & 0 \\ 0 & 0.50 & \sin(t) \\ 0 & 0 & 0.7 \end{bmatrix},$$
$$B_t = \begin{bmatrix} 0 & 0.3\sin(t) & 1 \end{bmatrix}^T,$$
$$C_t = \begin{bmatrix} 0 & 0.1 & 0.8 \end{bmatrix}.$$

The iteration length is set to be $N = 100$. The tracking reference is $y_d(t) = 0.5\sin\left(\frac{t}{20}\pi\right) + 0.25\sin\left(\frac{t}{10}\pi\right)$. The initial state for all the iterations is set to $x_k(0) = x_d(0) = 0$. The algorithm is run for 150 iterations for each case.

It should be noted that the actual tracking performance and the convergence speed depend on the average DDR. In the RSM, the assumption is made according to the worst case of successive data dropouts rather than the average DDR. In other words, a larger integer K does not necessarily imply a larger DDR. In the BVM, the expectation of the random variable $\gamma_k(t)$ corresponds to the average DDR. In the MCM, the DDR is jointly determined by the transition probability matrix, that is, it can be computed by deriving the stationary distribution. Specifically, the 3 models are simulated as follows.

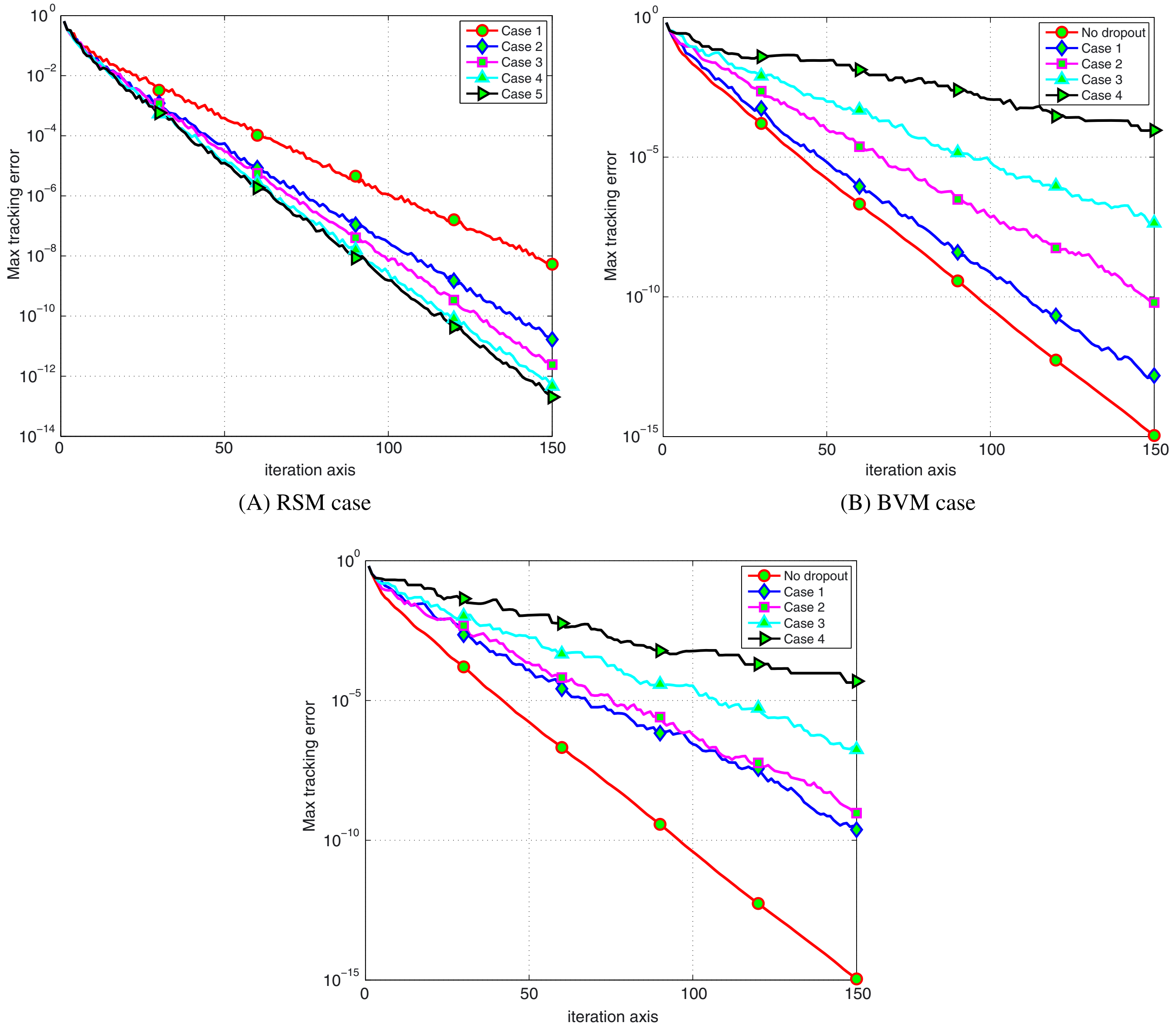
RSM We consider 5 cases for the RSM case. To simulate the data missing, we first separate the iterations into groups of M successive iterations, $M = 2, \cdots, 6$, that is, the iterations are separated as $\{kM+1, kM+2, \cdots, (k+1)M\}$, $k = 0, 1, 2, \cdots$, and randomly select 1 iteration from each group denoting the one whose data are dropped during transmission. In such case, the successive number K is 3. For example, take $M = 3$, then the iterations are separated as $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\cdots$, and from each group, 1 iteration is selected randomly. Therefore, the DDR for the above 5 cases is equal to $1/2$, $1/3$, $1/4$, $1/5$, and $1/6$, respectively.

BVM We consider 4 cases for the mathematical expectation of the random variable $\gamma_k(t)$. That is, $\bar{\gamma} = 0.9$, 0.7, 0.5, and 0.3. The smaller the expectation is, the larger the DDR is. Specifically, the DDR is equal to $1 - \bar{\gamma}$. As a consequence, the DDR values for the above 4 cases are 0.1, 0.3, 0.5, and 0.7. Additionally, the no-data-dropout case, namely, $\bar{\gamma} = 1$, is also simulated for a comparison.

MCM We consider 4 cases for the transition probability matrix as follows: $\mu = 0.8, \nu = 0.5$; $\mu = 0.7, \nu = 0.5$; $\mu = 0.6, \nu = 0.6$; $\mu = 0.5, \nu = 0.7$. The stationary distribution π for a transition probability matrix P can be computed from $\pi P = \pi$ and is given as $\pi = \left[\frac{1-\nu}{2-\mu-\nu}, \frac{1-\mu}{2-\mu-\nu}\right]$. Thus, the average DDR is equal to $\frac{1-\mu}{2-\mu-\nu}$. As a consequence, the DDR values for the above 4 cases are $2/7$, $3/8$, $1/2$, and $5/8$. Additionally, the no-data-dropout case, namely, $\mu = 1$ and $\nu = 0$, is also simulated for a comparison.

We first check the noise-free system case. In this case, we set $\sigma = 0.4$ and $L_t = 1$. The simulation is run according to the 3 data dropout models. The maximal tracking error for each iteration is defined as $\max_{1\le j\le N}|e_k(j)|$. The maximal tracking error profiles along the iteration axis are plotted in Figure 3. The Figure exhibits 2 observations. One is that the convergence speed slows down as the DDR increases, that is, larger DDR would result in slower convergence speed. The other one is that the maximal tracking error profiles approximate straight lines in the logarithm axis, which demonstrates that the convergence is exponential when no noise is involved in the system.

When the system is involved with random noise, an additional decreasing learning sequence should be introduced to the ILC rule to guarantee stable convergence of the proposed algorithms. The tracking performance is shown in Figure 4, where the random noise is assumed to be white Gaussian noise, namely, subject to $\mathcal{N}(0, \sigma^2)$ with $\sigma = 0.1$. In the simulation, the learning gain is set as $L_t = 1.5$, and the decreasing sequence selects $a_k = \frac{1}{k+1}$. We have some observations from Figure 4. First of all, due to the existence of random noise, the final tracking error cannot reduce to zero as the iteration number increases, and the maximal tracking error profiles would fluctuate heavily. Moreover, the introduction of $\{a_k\}$ makes the convergence speed much slower than in the noise-free case. However, it is a natural requirement for the control of stochastic systems. In addition, the influence of DDR on the convergence speed is similar to that of the noise-free case, which implies that stochastic noise and random data dropouts impact the performance independently.



(A) RSM case

(B) BVM case

(C) MCM case

FIGURE 3 Maximal tracking error profiles for the noise-free system along the iteration axis. A, Random sequence model (RSM) case where Cases 1 to 5 correspond to the data dropout rate (DDR) being $1/2$, $1/3$, $1/4$, $1/5$, and $1/6$, respectively; B, Bernoulli variable model (BVM) case where Cases 1 to 4 correspond to the DDR being 0.1, 0.3, 0.5, and 0.7, respectively; C, Markov chain model (MCM) case where Cases 1 to 4 correspond to the DDR being $2/7$, $3/8$, $1/2$, and $5/8$, respectively [Colour figure can be viewed at wileyonlinelibrary.com]

Comparing Figures 3 and 4, we can observe the connections and differences of the tracking performance between the 2 cases. On one hand, the convergence speed is determined by the average DDR for both cases as DDR is a direct index for the renewal frequency. On the other hand, the tracking precision depends much on the DDRs in the noise-free system case, whereas such dependence is not distinct for the stochastic system case because the stochastic noise will dominate the tracking error after several first iterations. In addition, the convergence speed for the stochastic system case greatly slows down due to the introduction of the decreasing gain sequence.

To sum up, the simulation results verify the theoretical results given in previous sections. Moreover, the convergence speed is determined by the selection of learning gain matrices as well as the DDR, where the former is a tunable factor, and the latter is an external factor due to the transmission quality of the channels. This paper is devoted to establishing a general convergence analysis framework for ILC under various data dropout models; thus, we mainly employ the basic simulations to show a validation of the theoretical results.

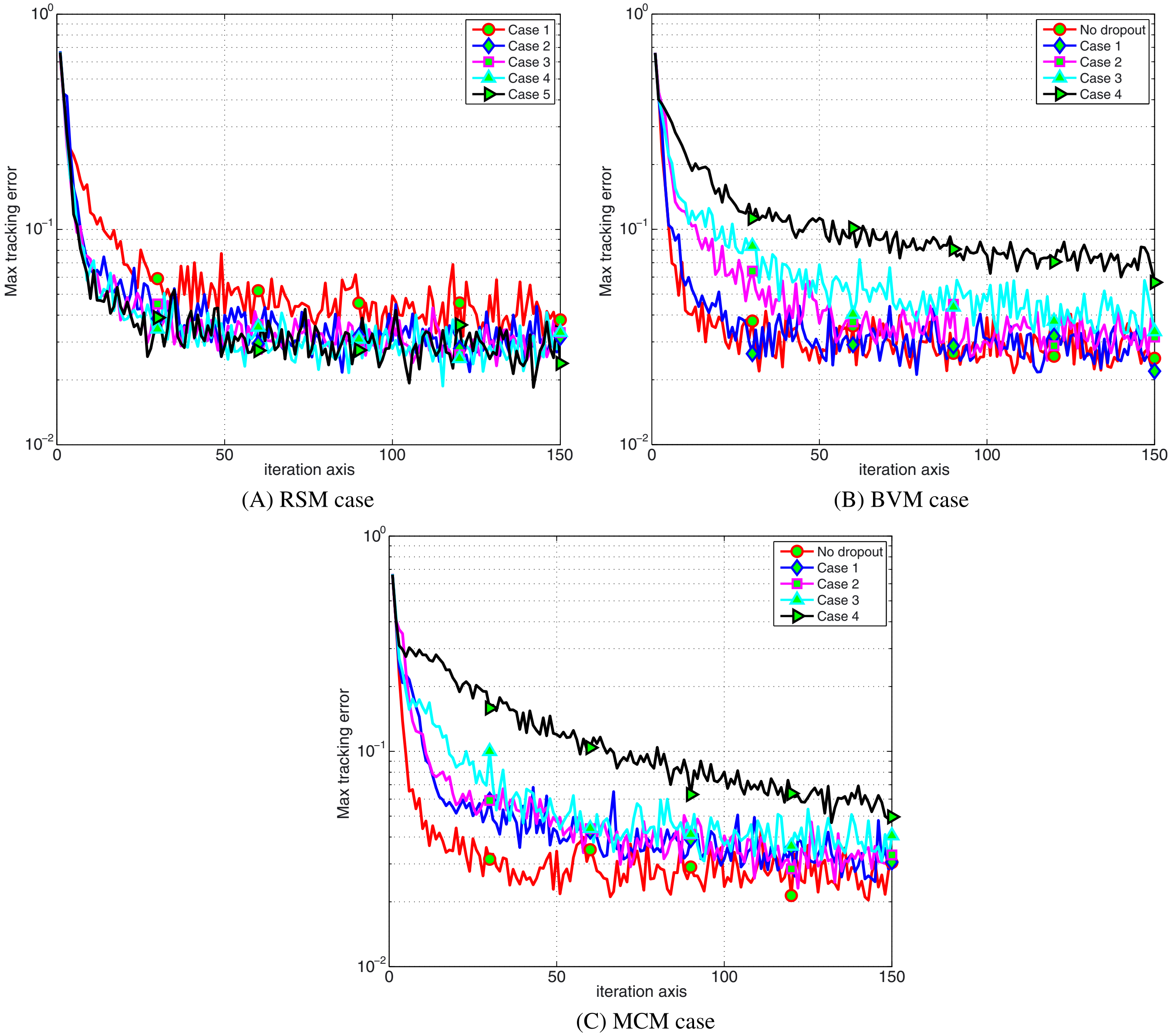


FIGURE 4 Maximal tracking error profiles for the noised system along the iteration axis. A, Random sequence model (RSM) case where Cases 1 to 5 correspond to the data dropout rate (DDR) being 1/2, 1/3, 1/4, 1/5, and 1/6, respectively; B, Bernoulli variable model (BVM) case where Cases 1 to 4 correspond to the DDR being 0.1, 0.3, 0.5, and 0.7, respectively; C, Markov chain model (MCM) case where Cases 1 to 4 correspond to the DDR being 2/7, 3/8, 1/2, and 5/8, respectively [Colour figure can be viewed at wileyonlinelibrary.com]

7 | CONCLUSIONS

In this paper, we have considered the convergence analysis for ILC under random data dropout environments. To this end, a framework was given to demonstrate both mean square and almost sure convergence properties of the classic P-type ILC update law for 3 kinds of data dropout models. Specifically, the RSM, the BVM, and the MCM were addressed in turn for both noise-free systems and stochastic systems, respectively. While we dealt with the case that the network at the measurement side suffers random data dropouts to clarify our idea, the extension to the case that the networks at both sides suffer random data dropouts directly follows the same analysis framework. In addition, the results can be extended to other types of ILC algorithms such as PD-type and current-iteration-feedback-integrated type update laws. For further research, we find that the transmission of data through networks would suffer many problems such as transmission error, bandwidth limitation, and transmission delay; therefore, it is of great interest to investigate on how to generalize the proposed results to deal with more general networked ILC problems.

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APPENDIX

Proof of Lemma 1

From Equation 15, we have

$$\vartheta_{k+1} \le (1 - d_1 a_k + d_2 a_k^2)\vartheta_k + d_2 d_3 a_k^2. \tag{A1}$$

Since $a_k \to 0$, we can choose a sufficient large integer k_0 such that $1 - d_1 a_k + d_2 a_k^2 < 1$ for all $k \ge k_0$, and then, we have

$$\vartheta_{k+1} \le \xi_k + d_4 a_k^2, \tag{A2}$$

where $d_4 \triangleq d_2 d_3$. As a result, it follows from Equation A2 and $\sum_{k=1}^{\infty} a_k^2 < \infty$ that $\sup_k \vartheta_k < \infty$, and then, ϑ_k converges. Based on this boundedness, from Equation A1, we have that

$$\vartheta_{k+1} \le (1 - d_1 a_k)\xi_k + d_5 a_k^2, \tag{A3}$$

where $d_5 > 0$ is a suitable constant. Noticing that $\sum_{k=1}^{\infty} a_k = \infty$ and $\sum_{k=1}^{\infty} a_k^2 < \infty$, we conclude that $\lim_{k\to\infty} \vartheta_k = 0$. □