Systems & Control Letters 107 (2017) 9–16

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Two novel iterative learning control schemes for systems with randomly varying trial lengths☆

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ARTICLE INFO

Article history:
Received 21 December 2016
Received in revised form 30 June 2017
Accepted 3 July 2017

Keywords:
Iterative learning control
Learning convergence
Randomly varying trial length

ABSTRACT

This paper proposes two novel improved iterative learning control (ILC) schemes for systems with randomly varying trial lengths. Different from the existing works on ILC with variable trial lengths that advocate to replace the missing control information by zero, the proposed learning algorithms are equipped with a searching mechanism to collect useful but avoid redundant past tracking information, which could expedite the learning speed. The searching mechanism is realized by the newly defined stochastic variables and an iteratively-moving-average operator. The convergence of the proposed learning schemes is strictly proved based on the contraction mapping methodology. Two illustrative examples are provided to show the superiorities of the proposed approaches.

1. Introduction

In our daily lives, one could complete a given task and improve the performance gradually provided that the operation is repeated. Such process for human being is usually called the learning process. Inspired by this basic cognition, iterative learning control (ILC) theory is developed for systems that are able to complete tasks over a fixed time interval and perform them repeatedly. By synthesizing the control input from the previous control input and tracking error, the controller is able to learn from the past experience and improve the current tracking performance. Since first introduced by Arimoto in 1980s [1], ILC has attracted much attention from both scholars and engineers over the past three decades and many achievements have been made [2–13].

When considering learning, a basic premise is that the desired task should be performed under same conditions such as identical initial condition and identical trial length for all iterations. In fact, such premise has been assumed in most ILC literature. However, one may find that this assumption is commonly violated in many practical applications due to system uncertainties. That is, the trial lengths may vary in the iteration domain. For instance, [14–17] provided several practical systems that run repeatedly but the trial lengths are not identical due to the complex external environments. Specifically, [14] investigated the application of ILC to humanoid robots, where the gaits problems were divided into phases defined by foot strike times and the durations of the phases were usually not the same from iteration to iteration during the learning process. Moreover, two biomedical systems including functional electrical stimulation for upper limb movement and for gait assistance were introduced in [15–17]. Due to the unknown dynamics and related complex factors, the learning process might end earlier and start the next iteration. Another example is the trajectory tracking with output constraints on a lab-scale gantry crane given in [18]. When the output constraints were violated, the load was wound up and the trial was terminated, which results in variable pass lengths for ILC [18]. Motivated by these observations, ILC problem with iteration-varying trial lengths has attracted more and more attention in recent years.

In the existing literature, there are some works addressing ILC design problems with non-uniform trial lengths from different technical perspectives [16–24]. First, Li et al. proposed an ILC framework for both discrete-time linear and continuous-time nonlinear systems with randomly varying trial lengths by introducing a stochastic variable to describe the randomness of trial lengths in [19] and [21], respectively. In [19], to deal with the randomly varying trial lengths, an iteration-average operator of all historical data was employed in the ILC algorithm to reduce the effect of the lost tracking information. While in [21], instead of using all historical control information, an iteratively-moving-average operator is adopted in ILC law where only the most recent control information will be utilized for learning since 'older' control information would reduce the corrective action from the most recent trials. Moreover, to avoid the utilization of λ-norm, a lifted framework of ILC for discrete-time linear systems was provided

☆ This work is supported by National Natural Science Foundation of China (61673045, 61304085) and Beijing Natural Science Foundation (4152040)

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http://dx.doi.org/10.1016/j.sysconle.2017.07.003
0167-6911/

in [20]. However, it is worthy noting that the convergence of the tracking errors in [19–21] is derived in the sense of mathematical expectation. Fortunately, there are some works showing stronger convergence properties of ILC with non-uniform trial lengths. For example, the almost sure and mean square convergence of a P-type ILC was established in [23,24]. Specifically, [23] considered a discrete linear system where the path statistic properties of the input error, namely, mathematical expectations and covariances, were first recursively calculated along the iteration axis. Based on the recursions of expectations and covariances of the input error, the convergence in the sense of expectation, mean square, and almost sure was derived in sequence. In [24], the ILC design problem was extended to a class of affine nonlinear systems, where the techniques used in [23] were no longer applicable for nonlinear systems. Thus, a modified λ-norm and a technical lemma were introduced to pave the way of showing the almost sure convergence of the tracking error. Furthermore, Seel et al. also contributed much on this topic, where the main focus lies in the monotonic convergence property [15,18,22] and practical applications [16,17]. A primary result is given in [15], where the authors presented the conditions of learning gain matrix for ensuring monotonic convergence. However, the calculations of the learning gain rely upon a completely known system model, which restricts the applicability of the proposed algorithm. A similar technique was applied to the trajectory tracking problem of a lab-scale gantry crane in [18]. The extended version of monotonic convergence with more detailed explanations was reported in a recent paper [22]. Additionally, [16,17] apply ILC with variable pass lengths in the Functional Electrical Stimulation (FES)-based treatment systems for strike patients. These two works also show that the addressed problem has great significance in real-time applications. However, it is worthwhile to highlight that a common feature of the works [18–24] on ILC with non-uniform trial lengths is to replace the missing tracking error information with zero. That is, when the tracking information is not available due to varying trial lengths, the lacked data is set to be zero. Therefore, how to develop a new ILC algorithm that is able to improve the control performance for systems with iteration-varying trial lengths, is an interesting and challenging problem.

Motivated by the above observations, in this paper, two novel improved ILC schemes are proposed for a class of discrete-time linear systems with randomly varying trial lengths. Different from the previous works on ILC with variable trial lengths that advocate to replace the missing tracking information by zero, the proposed learning algorithms are equipped with a searching mechanism to collect useful but avoid redundant past control information, which could expedite the learning speed. The searching mechanism is realized by introducing a new stochastic variable and an iteratively-moving-average operator.

The aim and main contribution of this paper is to reduce the impact of the randomly varying trial lengths to the learning control algorithm and to expedite the convergence speed. To achieve the objective, two ILC laws are proposed. More precisely, the first ILC scheme is proposed to reduce the redundant control information, which appears in the design of ILC laws in [19–21], while the second one is developed to make full use of the effective previous control information to further expedite the learning speed. In addition, the almost sure convergence for both ILC schemes is provided in a rigorous way.

The rest of the paper is organized as follows. Section 2 presents the problem formulation. Section 3 and 4 contribute to the controller design and convergence analysis. Furthermore, numerical simulations are given in Section 5 to verify the validation of the proposed control algorithms. Section 6 draws a conclusion of this work.

Notations. $\mathbf{R}$ is the real set and $\mathbf{R}^n$ is the n-dimensional space. $\mathbf{N}$ is the set of positive integers. $\|\cdot\|$ denotes the Euclidean norm of its indicated vector or matrix. Denote $\|\mathbf{f}(t)\|_\lambda \triangleq \sup_{t\in\{0,1,2,\ldots,T\}}\alpha^{-\lambda t}\|\mathbf{f}(t)\|$ and $\|\mathbf{f}(t)\|_s \triangleq \sup_{t\in\{0,1,2,\ldots,T\}}\|\mathbf{f}(t)\|$ the λ-norm and s-norm of a vector function $\mathbf{f}(t)$ respectively with $\lambda > 0$ and $\alpha > 1$.

2. Problem formulation

Consider the following discrete-time linear system

$$\begin{aligned} \boldsymbol{x}_k(t+1) &= A\boldsymbol{x}_k(t) + B\boldsymbol{u}_k(t),\\ \boldsymbol{y}_k(t) &= C\boldsymbol{x}_k(t), \end{aligned} \tag{1}$$

where $k \in \mathbf{N}$ is the iteration index, $t \in \{0, 1, 2, \ldots, T_k\}$ denotes the time instant, and T_k is the trial length at the kth iteration. Moreover, $\boldsymbol{x}_k(t) \in \mathbf{R}^n$, $\boldsymbol{u}_k(t) \in \mathbf{R}^p$, and $\boldsymbol{y}_k(t) \in \mathbf{R}^r$ denote the state, input, and output of the system (1), respectively. Furthermore, A, B and C are constant matrices with appropriate dimensions. It is worth to point out that the results and convergence analysis in this paper can be extended to linear time-varying systems straightforwardly, and thus we just consider the time-invariant case to clarify our idea. Let $\boldsymbol{y}_d(t), t \in \{0, 1, 2, \ldots, T_d\}$ be the desired output trajectory. Assume that, for any realizable output trajectory $\boldsymbol{y}_d(t)$, there exists a unique control input $\boldsymbol{u}_d(t) \in \mathbf{R}^p$ such that

$$\begin{aligned} \boldsymbol{x}_d(t+1) &= A\boldsymbol{x}_d(t) + B\boldsymbol{u}_d(t),\\ \boldsymbol{y}_d(t) &= C\boldsymbol{x}_d(t), \end{aligned} \tag{2}$$

where $\boldsymbol{u}_d(t)$ is uniformly bounded for all $t \in \{0, 1, 2, \ldots, T_d\}$ with T_d being the desired trial length.

The control objective is to tack the desired trajectory $\boldsymbol{y}_d(t)$, $t \in \{0, 1, 2, \ldots, T_d\}$ by determining a sequence of control inputs $\boldsymbol{u}_k$ such that the tracking error converges as the iteration number k increases.

Before addressing the controller design problem, the following assumptions are imposed.

A 1. *The coupling matrix CB is of full-column rank.*

A 2. *The initial states satisfy* $\|\boldsymbol{x}_d(0) - \boldsymbol{x}_k(0)\| \le \epsilon, \epsilon > 0$.

Remark 1. The initial state resetting problem is one of the fundamental issues in ILC field as it is a standard assumption to ensure the perfect tracking performance. In the past three decades, some papers have devoted to remove this condition by developing additional control mechanisms such as [25–27]. Under assumption A2, since the initial state is different from the desired initial state, it is impossible to achieve the perfect tracking. Instead, the ILC algorithms should force the system output to be as close as possible to the target.

Remark 2. It is worthy noting that, unlike the classic ILC theory that requires control tasks to repeat on a fixed time interval, the trial lengths T_k, $k \in \mathbf{N}$ are iteration-varying and may be different from the desired trial length T_d. For the case that the kth trial length is shorter than the desired trial length, both the system output and the tracking error information will be missing and cannot be used for learning. Thus, this paper aims to re-design ILC schemes to make up the missing signals by making full use of the previous available tracking information, and thus expedite the learning speed. Although some previous works have been published [19–21], a basic assumption is that the probability distribution of T_k is known prior. In this paper, the proposed ILC algorithms will be equipped with an automatic searching mechanism, thus the probability distribution of randomly varying trial lengths is no longer required.

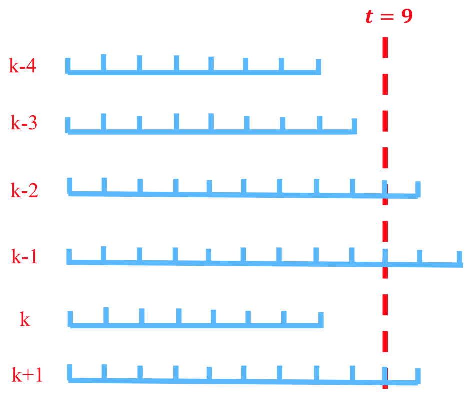

Fig. 1. Illustration of $S_{t,k}^m$: set $m=5$ and $t=9$, then $S_{t,k}^m=\{T_k,T_{k-3},T_{k-4}\}$, which implies that $n_t^k=3$ and $m-n_t^k=2$.

3. Controller design I and convergence analysis

In this section, a novel ILC algorithm will be developed to reduce the effect of redundant tracking information in the ILC algorithms in [19,21] and thus could expedite the convergence speed.

Recall that T_k is the trial length at kth iteration. It varies in the iteration domain randomly. Denote

$$S_{t,k}^m \triangleq \{T_{k+1-j} \mid t > T_{k+1-j},\ j=1,2,\ldots,m\}$$

where $m>1$ is an integer. Let $n_t^k=|S_{t,k}^m|$ be the amount of elements in the set $S_{t,k}^m$. That is, for a given time instant $t>0$ and a given iteration k, there are only $m-n_t^k$ iterations with the available tracking information in the past m iterations, and the tracking information at other iterations are missing. To show the definition of the set $S_{t,k}^m$ and the number n_t^k, a simple example is illustrated in Fig. 1. It is worthy pointing out that the number n_t^k is a random variable due to the randomness of the trial lengths. If we denote the probability of the occurrence of the output at time instant t as $p(t)$, the mathematical expectation of n_t^k can be calculated as $\mathbb{E}\{n_t^k\}=p(t)m$. Therefore, $m-n_t^k$ would increase to infinity as m goes to infinity in the sense of mathematical expectation. This property guarantees the reasonability of the following assumption.

A 3. *For a given iteration number k and a time instant $t\in\{0,1,\ldots,T_k\}$, the number $m-n_t^k\geq 1$. That is, there exists at least one iteration whose trial length is larger than t in the past m consecutive iterations.*

Remark 3. The assumption A3 is imposed to guarantee the learning effectiveness. If $m-n_t^k=0$, it gives $T_{k+1-j}<t$ for $j=1,2,\ldots,m$, namely, all trial lengths of the adjacent m iterations before the $(k+1)$th iteration are shorter than the given time t. This further means that at time instant t, there is no output information available and nothing can be learned from the past m iterations. By assuming $m-n_t^k\geq 1$, the effective learning process can be guaranteed. However, this assumption is not restrictive as $m-n_t^k$ would increase to infinity as m goes to infinity. In addition, assumption A3 implies that the iteration-varying trial lengths are not totally stochastic in this section. This assumption would be further relaxed in Section 4.

Similar to [19–21], we introduce a stochastic variable $\gamma_k(t)$, $t\in\{0,1,\ldots,T_d\}$, satisfying Bernoulli distribution and taking binary values 0 and 1. The relationship $\gamma_k(t)=1$ represents the event that the control process can continue to the time instant t at the kth iteration, while $\gamma_k(t)=0$ denotes the event that the control process cannot continue to the time instant t.

Based on the notations above and A3, the first proposed ILC law is presented as follows

$$\text{(I)}:\quad \boldsymbol{u}_{k+1}(t)=\frac{1}{m-n_t^k}\sum_{j=1}^{m}\gamma_{k+1-j}(t)\boldsymbol{u}_{k+1-j}(t)+\frac{1}{m-n_t^k}\Gamma\sum_{j=1}^{m}\gamma_{k+1-j}(t)\boldsymbol{e}_{k+1-j}(t+1), \tag{3}$$

where Γ is the learning gain matrix to be determined, and $\boldsymbol{e}_k\triangleq\boldsymbol{y}_d-\boldsymbol{y}_k$ represents the tracking error.

Remark 4. From (3), we can see that the stochastic variable $\gamma_k(t)$ is also adopted in the control input part, i.e., the first term on the right-hand side of (3). It implies that if the trial length is shorter than the given time t, both the corresponding input and tracking error signals will not be involved in the updating. The major difference between ILC law (3) and the ones in [19,21] lies in that the average operator will not incorporated redundant control information into the learning law, and thus the convergence speed could be expedited.

Remark 5. The initial m input and tracking error signals in the searching algorithm can be determined by using other control methods such as the classic feedback control that can stabilize the controlled system, and they will not affect the final convergence performance.

The convergence of the proposed ILC scheme (3) can be summarized in the following theorem.

Theorem 1. *Consider the system (1) and the ILC law (3). Assume that A1-A3 hold. If the following condition holds,*

$$\|I-\Gamma CB\|\leq\rho<1, \tag{4}$$

then the tracking error $\boldsymbol{e}_k$ will converge to the $\delta\epsilon$-neighborhood of zero asymptotically in the sense of λ-norm as k goes to infinity, where $\delta>0$ is a suitable constant to be defined later.

Proof. Denote $\Delta\boldsymbol{x}_k=\boldsymbol{x}_d-\boldsymbol{x}_k$ the state error, $\Delta\boldsymbol{u}_k=\boldsymbol{u}_d-\boldsymbol{u}_k$ the input error, and $\boldsymbol{e}_k=\boldsymbol{y}_d-\boldsymbol{y}_k$ the tracking error.

Subtracting both sides of the updating law (3) from $\boldsymbol{u}_d$, we have

$$\Delta\boldsymbol{u}_{k+1}(t)=\frac{1}{m-n_t^k}\sum_{j=1}^{m}\gamma_{k+1-j}(t)\Delta\boldsymbol{u}_{k+1-j}(t)-\frac{1}{m-n_t^k}\Gamma\sum_{j=1}^{m}\gamma_{k+1-j}(t)\boldsymbol{e}_{k+1-j}(t+1). \tag{5}$$

From (1) and (2), it gives

$$\boldsymbol{e}_k(t+1)=CA\Delta\boldsymbol{x}_k(t)+CB\Delta\boldsymbol{u}_k(t). \tag{6}$$

Substituting (6) into (5) implies

$$\begin{aligned}\Delta\boldsymbol{u}_{k+1}(t)=&\frac{1}{m-n_t^k}\sum_{j=1}^{m}\gamma_{k+1-j}(t)\Delta\boldsymbol{u}_{k+1-j}(t)\\&-\frac{1}{m-n_t^k}\Gamma\sum_{j=1}^{m}\gamma_{k+1-j}(t)\big[CA\Delta\boldsymbol{x}_{k+1-j}(t)\\&+CB\Delta\boldsymbol{u}_{k+1-j}(t)\big]\\=&\frac{1}{m-n_t^k}\sum_{j=1}^{m}\gamma_{k+1-j}(t)(I-\Gamma CB)\Delta\boldsymbol{u}_{k+1-j}(t)\\&-\frac{1}{m-n_t^k}\Gamma CA\sum_{j=1}^{m}\gamma_{k+1-j}(t)\Delta\boldsymbol{x}_{k+1-j}(t).\end{aligned} \tag{7}$$

Since $\Delta \boldsymbol{x}_k(t) = A^t \Delta \boldsymbol{x}_k(0) + \sum_{n=0}^{t-1} A^{t-n-1} B \Delta \boldsymbol{u}_k(n)$, it follows that

$$\begin{aligned}
\Delta \boldsymbol{u}_{k+1}(t) \\
&= \frac{1}{m-n_t^k}\sum_{j=1}^{m}\gamma_{k+1-j}(t)[I-\Gamma CB]\Delta \boldsymbol{u}_{k+1-j}(t) \\
&\quad - \frac{1}{m-n_t^k}\Gamma CA^{t+1}\sum_{j=1}^{m}\gamma_{k+1-j}(t)\Delta \boldsymbol{x}_{k+1-j}(0) \\
&\quad - \frac{1}{m-n_t^k}\Gamma CA\sum_{j=1}^{m}\gamma_{k+1-j}(t)\sum_{n=0}^{t-1}A^{t-n-1}B\Delta \boldsymbol{u}_{k+1-j}(n). \qquad (8)
\end{aligned}$$

Taking norm to both sides of (8), we can obtain that

$$\begin{aligned}
\|\Delta \boldsymbol{u}_{k+1}(t)\| \\
&\le \frac{1}{m-n_t^k}\sum_{j=1}^{m}\gamma_{k+1-j}(t)\|I-\Gamma CB\|\|\Delta \boldsymbol{u}_{k+1-j}(t)\| \\
&\quad + \kappa\alpha^{t+1}\epsilon \\
&\quad + \frac{1}{m-n_t^k}\kappa\beta\sum_{j=1}^{m}\gamma_{k+1-j}(t)\sum_{n=0}^{t-1}\alpha^{t-n}\|\Delta \boldsymbol{u}_{k+1-j}(n)\|, \qquad (9)
\end{aligned}$$

where $\frac{1}{m-n_t^k}\sum_{j=1}^{m}\gamma_{k+1-j}(t)=1$, $\alpha \ge \|A\|$, $\beta \ge \|B\|$, and $\kappa \ge \|\Gamma C\|$ are applied. Multiplying both sides of (9) by $\alpha^{-\lambda t}$, and taking the supremum with respect to the time t, we have

$$\begin{aligned}
\|\Delta \boldsymbol{u}_{k+1}(t)\|_\lambda &\le \frac{1}{m-n_t^k}\sum_{j=1}^{m}\gamma_{k+1-j}(t)\|I-\Gamma CB\|\|\Delta \boldsymbol{u}_{k+1-j}(t)\|_\lambda \\
&\quad + \kappa\alpha\epsilon + \frac{1}{m-n_t^k}\kappa\beta\sum_{j=1}^{m}\gamma_{k+1-j}(t) \\
&\quad \times \sup_t \alpha^{-\lambda t}\left(\sum_{n=0}^{t-1}\alpha^{t-n}\|\Delta \boldsymbol{u}_{k+1-j}(n)\|\right). \qquad (10)
\end{aligned}$$

Note that

$$\begin{aligned}
\sup_t \alpha^{-\lambda t}\left(\sum_{n=0}^{t-1}\alpha^{t-n}\|\Delta \boldsymbol{u}_{k+1-j}(n)\|\right) \\
&= \sup_t \alpha^{-(\lambda-1)t}\left(\sum_{n=0}^{t-1}\alpha^{-\lambda n}\|\Delta \boldsymbol{u}_{k+1-j}(n)\|\alpha^{(\lambda-1)n}\right) \\
&\le \sup_t \alpha^{-(\lambda-1)t}\left(\sum_{n=0}^{t-1}\sup_n(\alpha^{-\lambda n}\|\Delta \boldsymbol{u}_{k+1-j}(n)\|)\alpha^{(\lambda-1)n}\right) \\
&= \|\Delta \boldsymbol{u}_{k+1-j}(t)\|_\lambda \sup_t \alpha^{-(\lambda-1)t}\sum_{n=0}^{t-1}\alpha^{(\lambda-1)n} \\
&\le \frac{1-\alpha^{-(\lambda-1)T_d}}{\alpha^{\lambda-1}-1}\|\Delta \boldsymbol{u}_{k+1-j}(t)\|_\lambda, \qquad (11)
\end{aligned}$$

thus, (10) becomes

$$\begin{aligned}
\|\Delta \boldsymbol{u}_{k+1}(t)\|_\lambda &\le \frac{1}{m-n_t^k}\sum_{j=1}^{m}\gamma_{k+1-j}(t)\rho_0\|\Delta \boldsymbol{u}_{k+1-j}(t)\|_\lambda + \kappa\alpha\epsilon \\
&\le \rho_0 \max_{j=1,2,\ldots,m}\|\Delta \boldsymbol{u}_{k+1-j}(t)\|_\lambda + \kappa\alpha\epsilon, \qquad (12)
\end{aligned}$$

where $\rho_0 \triangleq \left(\|I-\Gamma CB\| + \frac{\kappa\beta(1-\alpha^{-(\lambda-1)T_d})}{\alpha^{\lambda-1}-1}\right)$, and the equation $\frac{1}{m-n_t^k}\sum_{j=1}^{m}\gamma_{k+1-j}(t)=1$ is applied.

Define $Q_{k+1} = \|\Delta \boldsymbol{u}_{k+1}(t)\|_\lambda - \frac{\kappa\alpha\epsilon}{1-\rho_0}$. From (12), it follows that

$$Q_{k+1} \le \rho_0 \max_{j=1,2,\ldots,m} Q_{k+1-j}. \qquad (13)$$

If $Q_{k+1} \le 0$, it means that $\|\Delta \boldsymbol{u}_{k+1}(t)\|_\lambda$ has entered the $\frac{\kappa\alpha\epsilon}{1-\rho_0}$-neighborhood of zero and will stay in the neighborhood. Thus, to show the bounded convergence, it is sufficient to analyze the scenario with $Q_{k+1} > 0$. Similar to (13), we have that $Q_{k+2} \le \rho_0 \max_{j=1,2,\ldots,m} Q_{k+2-j}$. Note that

$$\begin{aligned}
\max_{j=1,2,\ldots,m} Q_{k+2-j} &\le \max\{\max_{j=2,\ldots,m} Q_{k+2-j}, Q_{k+1}\} \\
&\le \max\{\max_{j=1,\ldots,m} Q_{k+1-j}, \rho_0 \max_{j=1,2,\ldots,m} Q_{k+1-j}\} \\
&= \max_{j=1,\ldots,m} Q_{k+1-j},
\end{aligned}$$

and it follows that $Q_{k+2} \le \rho_0 \max_{j=1,\ldots,m} Q_{k+1-j}$. By induction, we can obtain that $Q_{k+p} \le \rho_0 \max_{j=1,\ldots,m} Q_{k+1-j}$, $p = 1, 2, \ldots, m$. Therefore, it gives

$$\max_{p=1,\ldots,m} Q_{k+p} \le \rho_0 \max_{j=1,\ldots,m} Q_{k+1-j}, \qquad (14)$$

which implies the convergence of $\max_{p=1,\ldots,m} Q_{k+p}$, i.e., $\lim_{k\to\infty} \|\Delta \boldsymbol{u}(t)\|_\lambda = \frac{\kappa\alpha\epsilon}{1-\rho_0}$.

Moreover, $\boldsymbol{e}_k(t) = CA^t\Delta \boldsymbol{x}_k(0) + C\sum_{n=0}^{t-1}A^{t-n-1}B\Delta \boldsymbol{u}_k(n)$. Taking λ-norm on both sides of this equation gives

$$\|\boldsymbol{e}_k(t)\|_\lambda \le c\epsilon + c\beta\frac{1-\alpha^{-(\lambda-1)T_d}}{\alpha^\lambda - \alpha}\|\Delta \boldsymbol{u}_k(t)\|_\lambda, \qquad (15)$$

where $c = \|C\|$. Due to the convergence of $\|\Delta \boldsymbol{u}_k(t)\|_\lambda$, we can obtain the convergence of $\|\boldsymbol{e}_k(t)\|_\lambda$ to a neighborhood of zero where the bound is proportional to ϵ. In other words, there exists an appropriate $\delta > 0$ such that $\lim_{k\to\infty}\|\boldsymbol{e}_k(t)\|_\lambda \le \delta\epsilon$. This completes the proof. □

Remark 6. It is noted that the convergence condition given in Theorem 1, i.e., (4), is the same as the one in classic ILC, which is irrelevant with the probability distribution of the randomly varying trial lengths. This is one of the advantages of the proposed ILC scheme since it is shown that the same convergence condition can be applied to deal with more complex control problems. Although the probability distribution of the trial length is not involved in (4), different probability distribution will lead to different convergence speed. In details, for a given time instant t, the greater probability of the event $T_k \ge t$, the faster the convergence speed. This can be verified from (14). For a greater probability of $T_k \ge t$, we can select a smaller m, which indicates (14) will converge faster. However, due to the lack of analysis tools, currently it is difficult to present an analytic expression for the probability distribution and the convergence speed. This is an interesting problem, and should be addressed in future work.

Remark 7. The choice of m in the controller (3) depends on the length of the random interval for the trial length T_k. If the random interval is long, it implies that the trial length varies drastically in the iteration domain. In such a case, more previous trials will expedite the convergence speed because some of the missing information can be made up. While if the random interval is short, which means that the trial length in each iteration changes slightly and is close to the desired trial length, it is better to use a small number of previous trials. When the randomness is low, a large number of past trials may adversely weaken the learning effect because the large averaging operation would reduce the corrective action from the most recent trials.

4. Controller design II and convergence analysis

In this Section, we will develop a new ILC law to make full use of the previous control information, which thus could expedite the learning speed.

In order to facilitate the controller design, the following assumption is first imposed.

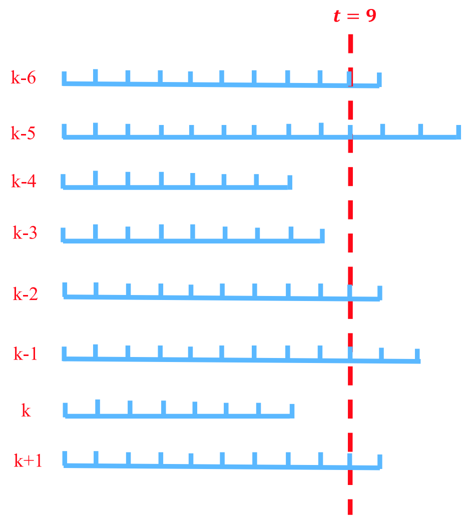

Fig. 2. Illustration of A3: set $m = 4$ and $t = 9$, then we can find that $T_j > t$, $j = k-1, k-2, k-5, k-6$, which implies that $r_{k,1} = 2, r_{k,2} = 3, r_{k,3} = 6, r_{k,4} = 7$.

A 4. *For a given iteration number $k > m$ and a time instant $t \in \{0, 1, \ldots, T_k\}$, we can find m past iterations such that $T_{k+1-r_{k,j}} > t$, $j = 1, 2, \ldots, m$, where $r_{k,j}, j = 1, 2, \ldots, m$ is an increasing sequence with $1 \le r_{k,j} \le k$ being an integer.*

Remark 8. The assumption A4 is reasonable since we can always find enough past iterations satisfying the assumption after a sufficiently large number of iterations. Otherwise, the learning process cannot be guaranteed. In practical, only the first few iterations may not satisfy A4. For these iterations, we can adopt the control algorithm in Section 3 or the ones in [19–21] if necessary, which will not affect the convergence of the learning algorithm. A simple example of A4 is illustrated in Fig. 2.

Based on A4, the second proposed ILC law is given as follows

$$\text{(II)}: \quad \boldsymbol{u}_{k+1}(t) = \frac{1}{m}\sum_{j=1}^{m} \boldsymbol{u}_{k+1-r_{k,j}}(t) + \frac{1}{m}\Gamma \sum_{j=1}^{m} \boldsymbol{e}_{k+1-r_{k,j}}(t+1). \quad (16)$$

Remark 9. From (16), it can be found that $r_{k,j}, j = 1, 2, \ldots, m$ are random variables because of the randomness of the trial lengths. The introduction of these random variables actually forms the searching mechanism in the control algorithm. By fully searching and utilizing the available tracking information, (16) is able to increase the convergence speed.

Remark 10. In this work, the ILC laws (3) and (16) are totally different. Based on A3, the searching mechanism in ILC law (3) is restricted in the last m iterations. Within the last m iterations, (3) incorporates all the available information into the controller. The main advantage for the controller (3) is some of 'too old' tracking information, which may weaken the correction from the latest iterations, can be avoided. However, the drawback is that the available historical information may be too scanty to improve the learning process if the probability of the occurrence of full trial length is small. While for the ILC law (16), it keeps searching until m available output signals are found. This controller is good at collecting all useful past control information, but the information far away from the currently iteration may degrade the learning performance. The comparison of these two ILC laws will presented in the numerical examples.

The second main result of this paper is summarized in the following theorem.

Theorem 2. *Consider system (1) and ILC law (16). Assume A1, A2, and A4 hold. If the following condition holds,*

$$\|I - \Gamma CB\| \le \rho < 1, \quad (17)$$

then the tracking error $\boldsymbol{e}_k$ will converge to the $\delta\epsilon$-neighborhood of zero asymptotically in the sense of λ-norm as k goes to infinity, where $\delta > 0$ is a suitable constant.

Proof. For a given time instant t, let $G_t \triangleq \{T_k \mid T_k > t, k = 1, 2, \ldots\}$. Define a new sequence $1 \le \sigma_1 < \sigma_2 < \cdots < \sigma_i < \cdots$ and assume T_{σ_i} is the ith elements of G_t. Then G_t can be represented as $G_t = \{T_{\sigma_1}, T_{\sigma_2}, \ldots, T_{\sigma_i}, \ldots\}$. For a given iteration number k, if $\sigma_i < k+1 \le \sigma_{i+1}$ and $i \ge m$, by the definition of G_t, the ILC law (16) can be rewritten as follows

$$\boldsymbol{u}_{k+1}(t) = \frac{1}{m}\sum_{j=1}^{m} \boldsymbol{u}_{\sigma_{i+1-j}}(t) + \frac{1}{m}\Gamma \sum_{j=1}^{m} \boldsymbol{e}_{\sigma_{i+1-j}}(t+1). \quad (18)$$

Moreover, the control input will not be updated from iteration $\sigma_i + 1$ to σ_{i+1}, namely,

$$\boldsymbol{u}_{\sigma_i+1}(t) = \cdots = \boldsymbol{u}_{k+1}(t) = \cdots = \boldsymbol{u}_{\sigma_{i+1}}(t). \quad (19)$$

Therefore, (18) and (19) imply that

$$\boldsymbol{u}_{\sigma_{i+1}}(t) = \frac{1}{m}\sum_{j=1}^{m} \boldsymbol{u}_{\sigma_{i+1-j}}(t) + \frac{1}{m}\Gamma \sum_{j=1}^{m} \boldsymbol{e}_{\sigma_{i+1-j}}(t+1). \quad (20)$$

Hence, it is sufficient to prove the convergence of the input sequence $\boldsymbol{u}_{\sigma_i}, i = 1, 2, \ldots$.

Similar to the proof of Theorem 1, the following inequality can be obtained

$$\|\Delta \boldsymbol{u}_{\sigma_{i+1}}(t)\|_\lambda \le \rho_0 \max_{j=1,2,\ldots,m} \|\Delta \boldsymbol{u}_{\sigma_{i+1-j}}(t)\|_\lambda + \kappa\alpha\epsilon. \quad (21)$$

Then following the same procedure as the latter part proof of Theorem 1, the convergence of the input sequence can be derived $\lim_{i\to\infty}\|\Delta \boldsymbol{u}_{\sigma_i}(t)\|_\lambda = \frac{\kappa\alpha\epsilon}{1-\rho_0}$, which further gives $\lim_{k\to\infty}\|\Delta \boldsymbol{u}_k(t)\|_\lambda = \frac{\kappa\alpha\epsilon}{1-\rho_0}$. Finally, the convergence of the tracking error can be proved similarly as the proof of Theorem 1, i.e., $\lim_{k\to\infty}\|\boldsymbol{e}_k(t)\|_\lambda \le \delta\epsilon$. The proof is thus completed. □

Remark 11. The learning algorithm (16) is stochastic due to the randomness of $r_{k,j}$. It seems that the algorithm (16) is deterministic but it is essentially stochastic because of the stochastic selection of suitable iterations, which can be seen from the subscripts of inputs and tracking errors. In addition, due to the introduction of randomly varying trial length, the convergence analysis in this paper uses a sequential contraction mapping as can be seen from (14) and (21). The major difference between the two recursions lies in that the sequential contraction in (14) is deterministic while in (21) is stochastic.

Remark 12. Similar results and convergence analysis can be extended to linear time-varying systems, namely, $A = A(t), B = B(t)$ and $C = C(t)$, and nonlinear systems with Lipschitz continuous uncertainties without significant efforts. For nonlinear systems without Lipschitz conditions, composite energy function (CEF) would be an optional approach. However, CEF-based ILC design with iteration-varying trial lengths is still an open problem.

Remark 13. In ILC field, 2D approach is another preferable analysis tools. For instance, in [28] 2D approach is applied to analyze the stability property of an inferential ILC, and in [29] a systematic procedure for ILC design by using 2D approach is developed. Therefore, investigating ILC with non-uniform trial lengths by 2D method would be an interesting research topic. Although there is no work reported on this topic in literature, it is not difficult to reformulate the problem addressed in this paper into 2D framework. Due to the variation of the trial lengths, the stochastic variable is still needed to modify the 2D variables when they are unavailable/missing. However, for nonlinear systems, it is difficult to apply 2D approach.

5. Illustrative example

In order to show the effectiveness and superiority of the proposed ILC schemes, the same discrete-time linear system (A, B, C) in [19] is considered, where

$$A = \begin{pmatrix} 0.50 & 0 & 1.00 \\ 0.15 & 0.30 & 0 \\ -0.75 & 0.25 & -0.25 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1.00 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 1.00 \end{pmatrix}.$$

Let the desired trajectory be $y_d(t) = \sin(2\pi t/50) + \sin(2\pi t/5) + \sin(50\pi t)$, $t \in I_d \triangleq \{0, 1, \ldots, 50\}$, and thus $T_d = 50$. Without loss of generality, set $u_0(t) = 0$, $t \in I_d$ in the first iteration. Moreover, assume that the trial length T_k varies from 30 to 50 satisfying discrete uniform distribution. This assumption is just a simple illustration. For other kinds of probability distribution, the proposed ILC scheme still works well since the probability distribution of trial lengths is not involved in the convergence conditions (4) and (17), and the influence of the probability distribution has also been discussed in Remark 6.

5.1. Simulations for ILC law (I)

In this subsection, let $m = 4$ and the learning gain is set as $L = 0.5$, which renders to $\|I - LCB\| = 0.5 < 1$. Firstly, we consider the case with identical initial condition, i.e., $\boldsymbol{x}_k(0) = [0, 0, 0]^T$, $k \in \mathbf{N}$. The performance of the maximal tracking error, $\|e_i\|_s \triangleq \sup_{t\in I_d} \|e_i\|$, is presented in Fig. 3. It shows that the maximal tracking error $\|e_i\|_s$ decreases from 1.945 to 0.0004 within 100 iterations. Meanwhile, to show the effectiveness of the proposed ILC scheme, the comparisons with the ILC law in [19] is also given in Fig. 3. It is obvious that by removing the redundant control input signal in the control laws, the proposed ILC scheme outperforms the one in [19]. In detail, it can be seen that the convergence of ILC law (I) is much faster and smoother, which could be more desirable to practical applications. It is noted that oscillations in the tracking error profiles in Fig. 3 are due to the variation of the trial lengths. The tracking performance of the ILC law (I) at different iterations is shown in Fig. 4, where we can see that after 50 iterations, the difference between y_{50} and y_d is almost invisible.

Furthermore, we consider the case with iteration-varying initial condition, namely, $\boldsymbol{x}_k(0) = \epsilon_k[1, 1, 1]^T$, $k \in \mathbf{N}$ with $\epsilon_k = 0.05\sin(k)$. The convergence of $\|e_i\|_s$ is given in Fig. 5. It is seen that after 50 iterations $\|e_i\|_s$ cannot be reduced further, and the final convergence bound is proportional to 0.05 which is the magnitude of $\boldsymbol{x}_k(0)$.

5.2. Simulations for ILC law (II)

This subsection will demonstrate the effectiveness of the proposed ILC scheme (II). Similar to Section 5.1, we select the learning gain $L = 0.5$. Let $m = 2$ and $\boldsymbol{x}_k(0) = [0, 0, 0]^T$, $k \in \mathbf{N}$. Fig. 6 shows the convergence of the maximal tracking error $\|e_i\|_s$. By comparing with the ILC law in [19], we can see that the proposed ILC algorithm (II) is able to expedite the convergence speed a hundredfold.

Moreover, Fig. 7 shows the comparison between the proposed ILC schemes (I) and (II). It is found that the ILC (I) presents a smoother tracking performance, while the ILC (II) wins by a faster convergence speed. The reason is that the ILC (II) incorporates more historic learning information into updating and expedites the convergence speed. However, utilizing some older control information in the algorithm may lead to oscillation in tracking performance. Therefore, which algorithm should be chosen is entirely dependent on the control targets. If the identical initial condition is not satisfied, e.g. $\boldsymbol{x}_k(0) = \epsilon_k[1, 1, 1]^T$, $k \in \mathbf{N}$, the ILC (16) still works well, as shown in Fig. 8, by sacrificing the convergence accuracy.

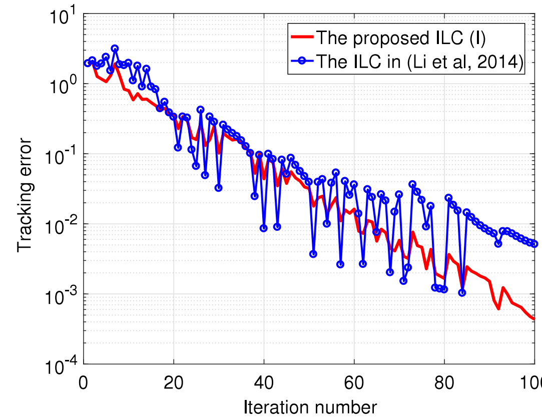

Fig. 3. The maximal tracking error profiles of the proposed ILC scheme (I) and the one in [19] (i.e., (Li et al., 2014)) under identical initial condition.

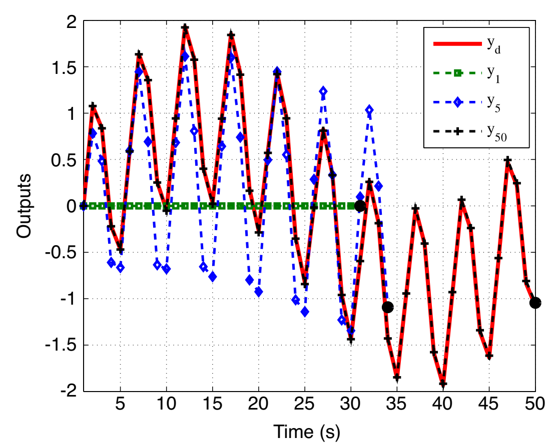

Fig. 4. The system outputs at the 1st, 5th and 50th iterations. The reference y_d is given for comparison.

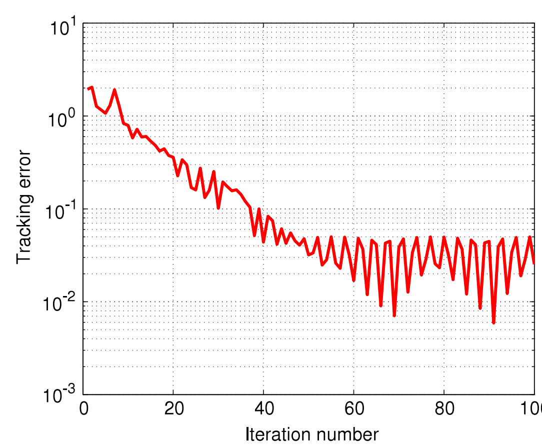

Fig. 5. The convergence of the maximal tracking error of ILC law (I) without identical initial condition.

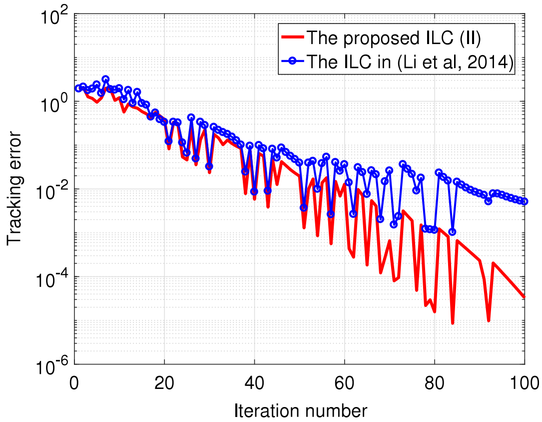

Fig. 6. The maximal tracking error profiles of the proposed ILC scheme (II) and the one in [19] (i.e., (Li et al., 2014)) under identical initial condition.

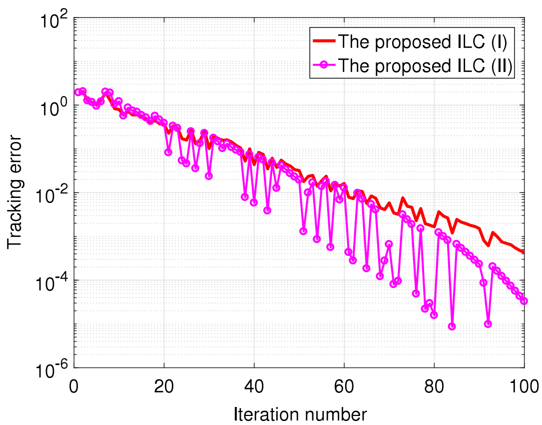

Fig. 7. The comparison of the proposed ILC algorithms (I) and (II).

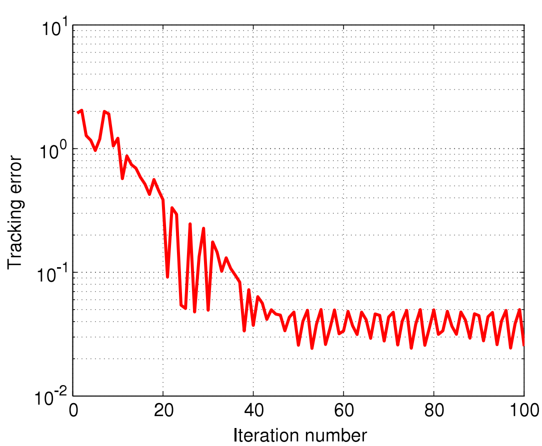

Fig. 8. The convergence of the maximal tracking error of ILC law (II) without identical initial condition.

6. Conclusion

This paper presents two novel improved ILC schemes for systems with randomly varying trial lengths. To improve the control performance under iteration-varying trial lengths, the proposed learning algorithms are equipped with a searching mechanism to collect useful but avoid redundant past control information, which is able to expedite the learning speed. The searching mechanism is realized by introducing newly defined stochastic variables and an iteratively-moving-average operator. The convergence of the proposed learning schemes is analyzed according to the contraction mapping methodology. Moreover, the efficiency of the proposed ILC schemes are verified by numerical examples. Extension to nonlinear systems and design framework of ILC with other imperfect learning conditions will be investigated in the next research phase.

References

[1] S. Arimoto, S. Kawamura, F. Miyazaki, Bettering operation of robots by learning, J. Robot. Syst. 1 (2) (1984) 123–140.
[2] D.A. Bristow, M. Tharayil, A.G. Alleyne, A survey of iterative learning control: a learning-based method for high-performance tracking control, IEEE Control Syst. Mag. 26 (3) (2006) 96–114.
[3] H.-S. Ahn, Y.Q. Chen, Kevin L. Moore, Iterative learning control: survey and categorization from 1998 to 2004, IEEE Trans. Syst. Man Cybern. C 37 (6) (2007) 1099–1121.
[4] D. Shen, Y. Wang, Survey on stochastic iterative learning control, J. Process Control 24 (12) (2014) 64–77.
[5] D. Huang, J.-X. Xu, V. Venkataramanan, T.C. Tuong Huynh, High performance tracking of piezoelectric positioning stage using current-cycle iterative learning control with gain scheduling, IEEE Trans. Ind. Electron. 61 (2) (2014) 1085–1098.
[6] D. Huang, J.-X. Xu, X. Li, C. Xu, M. Yu, D-type anticipatory iterative learning control for a class of inhomogeneous heat equations, Automatica 49 (2013) 2397–2408.
[7] X. Li, D. Huang, B. Chu, J.-X. Xu, Robust iterative learning control for systems with norm-bounded uncertainties, Internat. J. Robust Nonlinear Control 26 (2016) 697–718.
[8] D. Shen, Y. Xu, Iterative learning control for discrete-time stochastic systems with quantized information, IEEE/CAA J. Autom. Sin. 3 (1) (2014) 59–67.
[9] X. Bu, Z. Hou, S. Jin, R. Chi, An iterative learning control design approach for networked control systems with data dropouts, Internat. J. Robust Nonlinear Control 26 (2016) 91–109.
[10] D. Shen, J.-X. Xu, A novel Markov chain based ILC analysis for linear stochastic systems under general data dropouts environments, IEEE Trans. Autom. Control (2017). http://dx.doi.org/10.1109/TAC.2016.2638044.
[11] D. Meng, K.L. Moore, Learning to cooperate: networks of formation agents with switching topologies, Automatica 64 (2016) 278–293.
[12] T.D. Son, G. Pipeleers, J. Swevers, Robust monotonic convergent iterative learning control, IEEE Trans. Automat. Control 61 (4) (2016) 1063–1068.
[13] X. Li, Q. Ren, J.-X. Xu, Precise speed tracking control of a robotic fish via iterative learning control, IEEE Trans. Ind. Electron. 63 (4) (2016) 2221–2228.
[14] R.W. Longman, K.D. Mombaur, Investigating the use of iterative learning control and repetitive control to implement periodic gaits, Lecture Notes in Control and Inform. Sci. 340 (2014) 189–218.
[15] T. Seel, T. Schauer, J. Raisch, terative learning control for variable pass length systems, in: Proceedings of the 18th IFAC world congress, August 28–September 2, Milano, Italy, 2011, pp. 4880–4885.
[16] T. Seel, C. Werner, T. Schauer, The adaptive drop foot stimulator C Multivariable learning control of foot pitch and roll motion in paretic gait, Med. Eng. Phys. 38 (11) (2016) 1205–1213.
[17] T. Seel, C. Werner, J. Raisch, T. Schauer, Iterative learning control of a drop foot neuroprosthesis ł Generating physiological foot motion in paretic gait by automatic feedback control, Control Eng. Pract. 48 (2016) 87–97.
[18] M. Guth, T. Seel, J. Raisch, Iterative learning control with variable pass length applied to trajectory tracking on a crane with output constraints, in: Proceedings of the 52nd IEEE Conference on Decision and Control, Florence, Italy, 2013, pp. 6676–6681.
[19] X. Li, J.-X. Xu, D. Huang, An iterative learning control approach for linear systems with randomly varying trial lengths, IEEE Trans. Automat. Control 59 (7) (2014) 1954–1960.
[20] X. Li, J.-X. Xu, Lifted system framework for learning control with different trial lengths, Int. J. Autom. Comput. 12 (3) (2015) 273–280.
[21] X. Li, J.-X. Xu, D. Huang, Iterative learning control for nonlinear dynamic systems with randomly varying trial lengths, Internat. J. Adapt. Control Signal Process. 29 (11) (2015) 1341–1353.
[22] T. Seel, T. Schauer, J. Raisch, Monotonic convergence of iterative learning control systems with variable pass length, Internat. J. Control 90 (3) (2017) 393–406.
[23] D. Shen, W. Zhang, Y. Wang, C.-J. Chien, On almost sure and mean square convergence of p-type ILC under randomly varying iteration lengths, Automatica 63 (1) (2016) 359–365.

[24] D. Shen, W. Zhang, J.-X. Xu, Iterative learning control for discrete nonlinear systems with randomly iteration varying lengths, Systems Control Lett. 96 (2016) 81–87.
[25] Y. Chen, C. Wen, Z. Gong, M. Sun, An iterative learning controller with initial state learning, IEEE Trans. Automat. Control 44 (2) (1999) 371–375.
[26] M. Sun, D. Wang, Iterative learning control with initial rectifying action, Automatica 38 (7) (2002) 1177–1182.
[27] J.-X. Xu, R. Yan, On initial conditions in iterative learning control, IEEE Trans. Automat. Control 50 (9) (2005) 1349–1354.
[28] J. Bolder, T. Oomen, Inferential iterative learning control: A 2D-system approach, Automatica 71 (2016) 247–253.
[29] W. Paszke, E. Rogers, K. Galkowski, Experimentally verified generalized KYP Lemma based iterative learning control design, Control Eng. Pract. 53 (2016) 57–67.