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STOCHASTIC POINT-TO-POINT ITERATIVE LEARNING CONTROL BASED ON STOCHASTIC APPROXIMATION

Yun Xu, Dong Shen, and Xiao-Dong Zhang

ABSTRACT

An iterative learning control algorithm with iteration decreasing gain is proposed for stochastic point-to-point tracking systems. The almost sure convergence and asymptotic properties of the proposed recursive algorithm are strictly proved. The selection of learning gain matrix is given. An illustrative example shows the effectiveness and asymptotic trajectory properties of the proposed approach.

Key Words: Iterative learning control, linear stochastic systems, point-to-point control, stochastic approximation.

I. INTRODUCTION

It is readily apparent that performance of human motion tasks can be improved by repetition. This basic cognition motivates research into iterative learning control (ILC). ILC is a kind of optimization strategy that improves the tracking performance of a system which repeatedly completes some task over a fixed time interval. Many extensive studies have covered a large range of ILC topics, including the design of update laws, identical initialization conditions, robustness, optimization, transient behavior, and the combination of ILC with other control methods [1–9].

The standard ILC requires the system output to track the desired reference over the whole time interval [1–4]. However, in many practical applications such as 'pick and place' robotic task, satellite positioning, and production line automation, only partial reference needs to be accurately tracked while the left reference is with a large degree of freedom. This type of ILC is called point-to-point ILC. As a special case, if only the terminal point is required to track, it is termed the terminal ILC [10–12]. Great efforts have been made towards the point-to-point ILC problem. In [13,14], the problem was solved by iteratively updating the reference between trials instead of input profiles and the strict convergence analysis was provided. The paper [14] also presented an alternative method for the point-to-point problem, where the control input was linearly parameterized in term of basis functions that were constructed by system matrices and the parameters were updated according to specified tracking data. In addition, the continuous-time system case was addressed in [15] with a detailed comparison between experimental performance and theoretical results.

When considering a multiple input multiple output (MIMO) system, it is common that only some components of the output vector are required to satisfy certain conditions in practice. For example, consider a spatial motion control where the output position consists of three dimensions. We may impose a constraint only to the altitude but let the other two dimensions free. This type of general point-to-point tracking problem was studied in [16] and [17] for linear and nonlinear systems, respectively. The paper [16] provided an extensive formulation and analysis on gradient descent-based ILC and Newton method-based ILC with various mixed constraints. Readers can also refer to [18,19] for more experimental results.

However, in all the above studies, no stochastic noises are taken into account. This observation motivates us to further consider the stochastic point-to-point ILC problem. The objective of this paper is to address ILC for stochastic point-to-point tracking system with a general form of tracking reference as in [16,17]. Besides, we propose a stochastic approximation based solution to this new problem. It is worth pointing out that this paper is an extension of [20], where only the convergence was shown. While in this paper, a low computation algorithm is presented with a detailed analysis on convergence and asymptotical property. In addition, [21] also presented a gain-varying update method similar to our paper. However, the gain matrix in [21] was the inverse of the system model and the iteration varying gain was obtained by minimizing an objective function. While, in

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The authors are with College of Information Science and Technology, Beijing University of Chemical Technology, Beijing, China.

Dong Shen is the corresponding author (e-mail: shendong@mail.buct.edu.cn).

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this paper, we apply the stochastic approximation technique and propose a simplified algorithm without using inverse matrix.

The paper is arranged as follows. Section II provides the problem formulation; Section III gives the ILC algorithm associated with its convergence and asymptotical normality analysis; Section IV provides an illustrative example to show the effectiveness; some concluding remarks are given in Section V.

Notation. $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^m$ is the space of m-dimension vectors. The subscript T of a matrix denotes the transpose. $\mathcal{N}(0, Q)$ is the normal distribution with zero mean and covariance Q. For two sequences $\{a_k\}$ and $\{b_k\}$, we call $a_k = o(b_k)$ if $b_k \geq 0$ and $(a_k/b_k) \to 0$ as $k \to \infty$. $\|\cdot\|$ denotes the Euclidean norm of a vector or matrix.

II. PROBLEM FORMULATION

Consider the following LTV system:

$$\begin{aligned} x_k(t+1) &= A_t x_k(t) + B_t u_k(t) + w_k(t+1) \\ y_k(t) &= C_t x_k(t) + v_k(t) \end{aligned} \tag{1}$$

where the subscript k denotes different iterations, $k = 1, 2, \cdots$, and t denotes an arbitrary time in an iteration. Denote the length of one iteration as N. $x_k(t) \in \mathbb{R}^n$, $u_k(t) \in \mathbb{R}^p$, and $y_k(t) \in \mathbb{R}^q$ are system state vector, input vector and output vector, respectively. System matrices A_t, B_t and C_t are with appropriate dimensions. $w_k(t)$ and $v_k(t)$ are system noise and measurement noise, respectively. In this paper, the noise is simply assumed to be zero mean Gaussian white noise, *i.e.*, with normal distribution. Besides, for different iterations and different time instances, the noise signals are uncorrelated. For any t, both $\{w_k(t), k = 1, 2, \cdots\}$ and $\{v_k(t), k = 1, 2, \cdots\}$ are independent and identically distributed sequences.

Assume that $C_{t+1}B_t$ is of full row rank. This condition implies that the relative degree is one. The high relative degree is an important issue in the ILC field [22,23]. The results of this paper can be extended to the high relative degree case with slight modifications similar to [22,23]. The input and output can be lifted as super-vector forms $\boldsymbol{u}_k = [u_k^T(0), u_k^T(1), \cdots, u_k^T(N-1)]^T \in \mathbb{R}^{pN}$, $\boldsymbol{y}_k = [y_k^T(1), y_k^T(2), \cdots, y_k^T(N)]^T \in \mathbb{R}^{qN}$.

In addition, let

$$G = \begin{bmatrix} C^+B_0 & 0 & \cdots & 0 \\ \mathcal{M}_0^2 & C^+B_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{M}_0^N & \mathcal{M}_1^N & \cdots & C^+B_{N-1} \end{bmatrix}$$

where $C^+B_t \triangleq C_{t+1}B_t$, $\mathcal{A}_i^j \triangleq A_jA_{j-1}\cdots A_i$, $\mathcal{M}_i^j \triangleq C_j\mathcal{A}_{i+1}^{j-1}B_i$.

Then we can rewrite (1) as

$$\boldsymbol{y}_k = G\boldsymbol{u}_k + \boldsymbol{y}_k^0 + \epsilon_k \tag{2}$$

where $\boldsymbol{y}_k^0$ is the response to initial states, given by $\boldsymbol{y}_k^0 = [(C_1\mathcal{A}_0)^T, (C_2\mathcal{A}_0^1)^T, \cdots, (C_N\mathcal{A}_0^{N-1})^T]^T x_k(0)$, where $x_k(0)$ denotes the initial state. In this paper, two cases of $\boldsymbol{y}_k^0$ are discussed, namely, the identical initialization condition case and the asymptotic accuracy case. Without loss of any generality, it is simply assumed $\boldsymbol{y}_k^0 = 0$ for the former case and $\boldsymbol{y}_k^0 \to 0$ for the latter case. The stochastic noise term ϵ_k is expressed as $\epsilon_k = [(v_k(1)+C_1w_k(1))^T, \cdots, (v_k(N)+C_N\sum_{i=0}^{N-1}\mathcal{A}_0^{i-1}w_k(N-i))^T]^T$. Thus ϵ_k is assumed as zero-mean Gaussian process noise with covariance Q, *i.e.*, $\epsilon_k \sim \mathcal{N}(0, Q)$. Q is defined as $\mathbb{E}(\epsilon_k\epsilon_k^T)$, and thus depends on both noises and system matrices.

For standard ILC, the desired reference is

$$\boldsymbol{y}_d = [y_d^T(1), y_d^T(2), \cdots, y_d^T(N)]^T \in \mathbb{R}^{qN}. \tag{3}$$

Denote the standard tracking error as $\boldsymbol{e}_k = \boldsymbol{y}_d - \boldsymbol{y}_k$.

In many practical applications, only part of $\boldsymbol{y}_d$ rather than $\boldsymbol{y}_d$ itself is required to track. To this end, we give a variant modelling procedure of [16]. Suppose that only l_j components of the output at time j are required to track, $0 \leq l_j \leq q$, $j = 1, 2, \cdots, N$. If $l_j = 0$, it means that the output at time j is completely disregarded. If $l_j \neq 0$, let us denote the tracking components by $1 \leq n_{j,1} < n_{j,2} < \cdots < n_{j,l_j} \leq q$. Removing all the points that do not need to be followed from the original objective $\boldsymbol{y}_d$, a new reference trajectory $\boldsymbol{y}_r$ with dimension l is obtained where $l = \sum_{j=1}^N l_j$. That is, $\boldsymbol{y}_r$ is a condensed reference trajectory of $\boldsymbol{y}_d$.

Let us first define a row vector $\theta \in \mathbb{R}^{qN}$ with the same dimension of $\boldsymbol{y}_d$. If the ith component of $\boldsymbol{y}_d$ is required to track, then define the ith component of θ as one, *i.e.*, $\theta_i = 1$, otherwise define it as zero. That is

$$\theta_i = \begin{cases} 1, & \text{if the } i\text{th component is the target} \\ & 1 \leq i \leq qN \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Then construct a matrix $\Phi \in \mathbb{R}^{l \times qN}$ as follows

$$\Phi_{i,j} = \begin{cases} 1, & \text{if } \theta_j = 1 \text{ and } \sum_{k=1}^j \theta_k = i \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

for $i = 1, \cdots, l$, $j = 1, \cdots, qN$. Then, it is evident that Φ is of full row rank, *i.e.*, $rank(\Phi) = l$. Moreover, $\boldsymbol{y}_r$ and $\boldsymbol{y}_d$ satisfy the following relationship

$$\boldsymbol{y}_r = \Phi \boldsymbol{y}_d. \tag{6}$$

For deterministic systems, the control objective is to find an input sequence such that $\|\boldsymbol{e}_k\| \to 0$ as $k \to \infty$. However, this control objective is not suitable for the stochastic point-to-point tracking problem due to two reasons. The first one is that all components of $\boldsymbol{e}_k$ can not be available in applications. The other one is that the tracking errors cannot converge to zero because of the existence of stochastic noises. In this paper, we define $\Phi\hat{\boldsymbol{e}}_k \triangleq \boldsymbol{y}_r - \Phi G \boldsymbol{u}_k$. The control objective is to guarantee that $\Phi\hat{\boldsymbol{e}}_k \to 0$ as $k \to \infty$.

For further analysis, we have the following property on the matrix ΦG.

Lemma 1. Assume that C^+B_t is of full row rank, $\forall t$, then ΦG is of full row rank. That is, $rank(\Phi G) = l$.

Proof. Since C^+B_t is of full row rank, we have $rank(C^+B_t) = q$. From the definition of G, it is evident $rank(G) = qN$. By Sylvester's rank inequality, $rank(\Phi) + rank(G) - qN \le rank(\Phi G)$. This further yields that $l \le rank(\Phi G)$. On the other hand, $rank(\Phi G) \le \min\{rank(\Phi), rank(G)\} = l$. Thus, we have $rank(\Phi G) = l$.

III. ILC ALGORITHM AND ITS ANALYSIS

In this section, we design the ILC algorithm based on the stochastic approximation to generate an input sequence such that the control objective is achieved.

Let $\{a_k\}$ be a sequence satisfying that

$$\begin{aligned} &a_k > 0, \quad a_k \xrightarrow[k\to\infty]{} 0, \quad \sum_{k=1}^{\infty} a_k = \infty, \\ &\frac{a_k - a_{k+1}}{a_k a_{k+1}} \xrightarrow[k\to\infty]{} \alpha \ge 0. \end{aligned} \tag{7}$$

The ILC update law is now defined as

$$\boldsymbol{u}_{k+1} = \boldsymbol{u}_k + a_k L(\boldsymbol{y}_r - \Phi \boldsymbol{y}_k) \tag{8}$$

where L is the learning gain matrix to be specified later. The above update law actually is a stochastic approximation algorithm. In the following the convergence and asymptotic normality of (8) are thus proved based on the stochastic approximation results. To this end, two theorems from stochastic approximation theory will be used in the following analysis. For readability, both theorems are included in the appendix.

Remark 1. Comparing with the conventional P-type algorithm $\boldsymbol{u}_{k+1} = \boldsymbol{u}_k + L(\boldsymbol{y}_r - \Phi \boldsymbol{y}_k)$, it is seen that a decreasing gain a_k is added to (8). For deterministic systems, the conventional P-type algorithm can guarantee a satisfactory convergence behavior. While for stochastic systems, such algorithm may not behave well since the historical stochastic noises cannot be canceled by a fixed learning gain. That is, the input sequence fails to achieve a stable convergence due to the existence of stochastic noises. This is the first reason why we introduce a_k to (8). Moreover, the decreasing gain a_k could suppress stochastic noises efficiently. As a matter of fact, the decreasing gain is somewhat a necessary requirement for eliminating the influence of stochastic noises [24].

3.1 Convergence of the ILC Algorithm

Let us first consider the identical initialization condition case.

Theorem 1. For system (2) with initial value $\boldsymbol{y}_k^0 = 0$, design L such that $-\Phi G L$ is stable, then the ILC update law (8) with arbitrary initial input $\boldsymbol{u}_0$ would guarantee that $\Phi\hat{\boldsymbol{e}}_k \to 0$ as $k \to \infty$.

Proof. From (8), we have

$$\Phi\hat{\boldsymbol{e}}_{k+1} = \Phi\hat{\boldsymbol{e}}_k - a_k \Phi G L \Phi \hat{\boldsymbol{e}}_k + a_k \Phi G L \Phi \epsilon_k. \tag{9}$$

Notice that $-\Phi G L$ in (9) corresponds to the matrix H_k of Theorem 3 in the appendix, which is iteration-invariant and stable by the design of L.

Moreover, one could find that $\Phi G L \Phi \epsilon_k$ in (9) corresponds to μ_{k+1} of Theorem 3 in the appendix. Note that ϵ_k is independent along the iteration axis, and so is $\Phi\epsilon_k$, Therefore, it is obvious that $\sum_{k=1}^{\infty} \|a_k \Phi \epsilon_k\|^2 \le trace(\Phi Q \Phi^T) \sum_{k=1}^{\infty} a_k^2 < \infty$. By Khintchine-Kolmogorov convergence theorem, we further have that $\Phi G L \Phi \sum_{k=1}^{\infty} a_k \epsilon_k < \infty$. The proof is thus completed by directly applying Theorem 3.

Remark 2. Here, the stability of a matrix is defined as that all its eigenvalues are with negative real parts. Since $rank(\Phi G) = l$, the design of L could be implemented by solving a linear matrix inequality (LMI): $\Phi G L > 0$. This approach for L owns advantages on robustness [25]. However, such selection of L leads to a non-causal design of ILC. In the next subsection, more discussions on the selection of L are given.

In many applications, the identical initialization condition may not be satisfied. However, the initial state may converge to zero asymptotically, or we could introduce an initial state learning mechanism to make it so. In this case, we could formulate the initial condition as $\boldsymbol{y}_k^0 \to 0$. Then we have the following corollary.

Corollary 1. For system (2) with initial value $\boldsymbol{y}_k^0 \to 0$, design L such that $-\Phi GL$ is stable, then the ILC update law (8) with arbitrary initial input $\boldsymbol{u}_0$ would guarantee that $\Phi\hat{\boldsymbol{e}}_k \to 0$ as $k \to \infty$.

Proof. Notice that the initial value $\boldsymbol{y}_k^0 \neq 0$. Then, there is an extra term added to (9), which becomes

$$\begin{aligned}\Phi\hat{\boldsymbol{e}}_{k+1} &= \Phi\hat{\boldsymbol{e}}_k - a_k\Phi GL\Phi\hat{\boldsymbol{e}}_k \\ &\quad + a_k\Phi GL\Phi\epsilon_k + a_k\Phi GL\Phi\boldsymbol{y}_k^0\end{aligned} \tag{10}$$

Thus $\Phi GL\Phi\boldsymbol{y}_k^0$ corresponds to the notation ν_{k+1} of Theorem 3 in the appendix, leading that the condition on ν_k is obviously satisfied. Then the corollary holds obviously according to Theorem 3.

Remark 3. In both Theorem 1 and Corollary 1, it is claimed that the initial input can be arbitrarily given. This implies that the convergence is independent of the initial value of the algorithm (8). In practical applications, because we have little prior knowledge of system information, the initial input $\mathbf{u}_0$ is usually set to be zero. The influence of the initial input to the variance of tracking errors was discussed in [26] where the initial input was formulated as a random variable.

3.2 Selection of Learning Gain Matrix

In the last subsection, the design condition on learning gain matrix L is to make $-\Phi GL$ stable, *i.e.*, to ensure that all eigenvalues of ΦGL are with positive real parts. Obviously, there is a large selection range of L since ΦG is of full row rank.

The first and apparent selection is $L = (\Phi G)^T$, which results in that ΦGL is a positive definite matrix. The selection actually is the gradient algorithm, which has been adopted in [16,25]. However, on one hand, this selection requires the full knowledge of system matrices from the formulation of G. On the other hand, one may worry about the computation load when the iteration length N is large.

In order to overcome the above disadvantages, an alternative selection of L is given as follows. Note that G is a block lower triangular matrix with C^+B_t being its diagonal blocks. Thus these diagonal blocks would play the major part in control. Hence, we could form another block diagonal matrix as

$$G_1 = diag\{C^+B_0, \cdots, C^+B_{N-1}\}$$

and then we can select an alternative L as $L = (\Phi G_1)^T$.

In this case, GG_1 still is a block lower triangular matrix with diagonal block being $(C^+B_t)(C^+B_t)^T$. Since C^+B_t is of full row rank, the diagonal block $(C^+B_t)(C^+B_t)^T$ is thus positive definite. Hence, all eigenvalues of GG_1 are with positive real parts although it is not symmetric, and so are the eigenvalues of ΦGL. We can find that only the information of input-output coupling matrix C^+B_t is required. On the other hand, the coupling matrix C^+B_t denotes the control direction information, which is necessary when designing the controller. Moreover, the selection $L = (\Phi G_1)^T$ avoids the non-causality of controller design.

In addition, the computation load could be further reduced by decomposing the lifted update law (8) into non-lifted forms. In other words, the lifted update law (8) is used for analysis convenience. To be specific, we give an illustrative example based on the SISO LTI system, *i.e.*, $A_t \equiv A$, $B_t \equiv B$, $C_t \equiv C$. The entire output is of N dimension. Denote the tracking error as $\hat{e}_{1,k}, \cdots, \hat{e}_{N,k}$. We assume there are $\kappa \leq N$ outputs required to track and the locations are $j_1, \cdots, j_\kappa$. Let $j_0 = 0$. Then the update law along the time axis for $L = (\Phi G)^T$ case is given as

$$u_{k+1}(t) = \begin{cases} u_k(t) + a_k \sum_{m=i}^{\kappa} CA^{j_m-t-1}B\hat{e}_{j_m,k}, \\ \qquad\qquad j_{i-1} \leq t < j_i, 1 \leq i \leq \kappa \\ u_k(t), \qquad j_\kappa \leq t \leq N-1 \end{cases} \tag{11}$$

while for $L = (\Phi G_1)^T$ case the update law is given as

$$u_{k+1}(t) = \begin{cases} u_k(t) + a_k\ CB\hat{e}_{j_i,k}, \\ \qquad\qquad t = j_i - 1, 1 \leq i \leq \kappa \\ u_k(t), \quad \text{otherwise} \end{cases} \tag{12}$$

3.3 Asymptotic properties of the ILC algorithms

In this subsection, further asymptotic property analysis will be done for the ILC algorithms with a decreasing gain. We have proved that $\Phi\boldsymbol{e}_k \to 0$, which is a path-wise result. Noticing that the stochastic noise is involved in our system, thus $\Phi\boldsymbol{e}_k$ is a random variable. We will show that $\frac{1}{\sqrt{a_k}}\Phi\boldsymbol{e}_k$ is asymptotically normal, *i.e.*, the distribution of $\frac{1}{\sqrt{a_k}}\Phi\boldsymbol{e}_k$ converges to a normal distribution as $k \to \infty$. This could be regarded as a statistics result and is specified as follows.

Theorem 2. For system (2) with initial value $\boldsymbol{y}_k^0 = 0$, design L such that $-\Phi GL + \frac{\alpha}{2}I$ is stable, where α is defined by (7), then the ILC update law (8) with arbitrary initial input $\boldsymbol{u}_0$ would guarantee that $\frac{1}{\sqrt{a_k}}\Phi\boldsymbol{e}_k$ is asymptotically normal, *i.e.*,

$$\frac{1}{\sqrt{a_k}}\Phi\boldsymbol{e}_k \xrightarrow[k\to\infty]{d} \mathcal{N}(0, S) \tag{13}$$

where

$$S = \int_0^{\infty} e^{(-J+\frac{\alpha}{2}I)z} JQ'J^T e^{(-J^T+\frac{\alpha}{2}I)z} dz \tag{14}$$

Proof. Similar to Theorem 1, noticing the facts that $J = \Phi GL$ and Q' is the covariance of $\Phi\epsilon_k$, we have that the covariance of $\Phi GL\Phi\epsilon_k$ is $JQ'J^T$. The proof is completed by using Theorem 4.

For the case that the initial state can be asymptotically exactly reset, *i.e.*, $\boldsymbol{y}_k^0 \to 0$, the above asymptotic normal property is also valid, if the convergence speed of the initial value satisfies certain requirements. Thus the following corollary is presented.

Corollary 2. For system (2) with initial value $\boldsymbol{y}_k^0 \to 0$ satisfying $\|\boldsymbol{y}_k^0\| = o(\sqrt{a_k})$, design L such that $-\Phi GL + \frac{\alpha}{2}I$ is stable, where α is defined by (7), then the result of Theorem 2 still holds.

Proof. Using similar steps of Theorem 1 and Theorem 2, the result directly holds based on Theorem 4 in the appendix.

Remark 4. It is obvious that the conditions in (7) are all satisfied if $a_k = \frac{\beta}{k}$ with $\beta > 0$, and then $\alpha = \frac{1}{\beta}$. β would affect the convergence rate and normality of the algorithm in practical application. The conditions of a_k are essentially required by the stochastic approximation algorithm. Roughly speaking, rapid decreasing of a_k may result in fast convergence rate and large α would impose more limitations on design of L.

IV. ILLUSTRATIVE EXAMPLE

Consider a linear stochastic system with the following system matrices:

$$A_t = \begin{bmatrix} 0.5(1-\cos((t-1)\pi/10)) & -0.07 & -0.26 \\ -0.03 & 0.38 & -0.3 \\ -0.1 & -0.13 & 0.4 \end{bmatrix},$$

$$B_t = \begin{bmatrix} -1 & 0.1\sin((t-1)\pi/10) & 0 \\ 0 & -1 & 0.4 \\ 0.2 & 0 & -1 \end{bmatrix},$$

$$C_t = \begin{bmatrix} 1.8 & 0.5 & 0.3 \\ 0.4 & 1.6t^{-0.5} & 1.2 \end{bmatrix}.$$

To make the illustration simple and clear, let $N = 6$, then $\boldsymbol{y}_k \in \mathbb{R}^{12}$ and $\boldsymbol{u}_k \in \mathbb{R}^{18}$. The noise ϵ_k is assumed to be zero-mean Gaussian process noise with the covariance $Q = 0.06^2 I_{12}$, where I_n denotes n-dimensional unit matrix. Suppose the reference points are $y_d^{(1)}(1)$, $y_d^{(2)}(3)$, $y_d^{(1)}(4)$ and $y_d^{(1)}(6)$. The selected reference trajectory is $\boldsymbol{y}_r = [1\ 2\ 1.5\ 1]^T$.

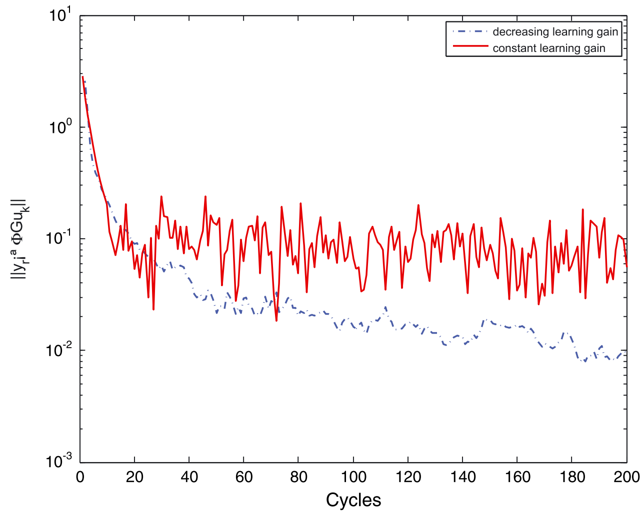

Fig. 1. Norm of the tracking error $\|\boldsymbol{y}_r - \Phi G\boldsymbol{u}_k\|$. [Color figure can be viewed at wileyonlinelibrary.com]

The parameters in (8) are set as $a_k = 1/k$ and $L = (\Phi G_1)^T$. The algorithm is run for 200 iterations. The norm of the modified tracking error $\|\Phi\hat{e}_k\|$ is presented in Fig. 1, labeled by decreasing learning gain and denoted by the dash-dot line. As one could see, the error reduces to zero rapidly. This further means that the actual tracking error will be caused only by the system and measurement noises of the current iteration asymptotically, while the latter noises cannot be canceled by any learning algorithm.

The decreasing learning gain a_k plays an important role to make a zero-convergence of the modified error $\Phi\hat{e}_k$. As a comparison, the conventional P-type update law is also simulated, where the gain a_k is fixed to be 0.2. The norm of its modified tracking error $\|\Phi\hat{e}_k\|$ is presented in Fig. 1, labeled by constant learning gain and denoted by the solid line.

As one can see, the constant learning gain may lead the modified tracking error to decrease a little faster in the first fewer iterations, but fluctuate with a larger amplitude since then. However, the decreasing learning gain could make the modified tracking error keep decreasing as the iteration number increases. This coincides with the fact that the constant learning gain fails to ensure zero-error convergence of the modified tracking error while the decreasing learning gain can.

In addition, one may argue whether different α defined in (7) would have significant influence on the convergence behavior. Noticing Remark 4, it is equivalent to show how the algorithm behaves for different β. To this end, three cases are selected, *i.e.*, $\beta = 1$, $\beta = 2$,

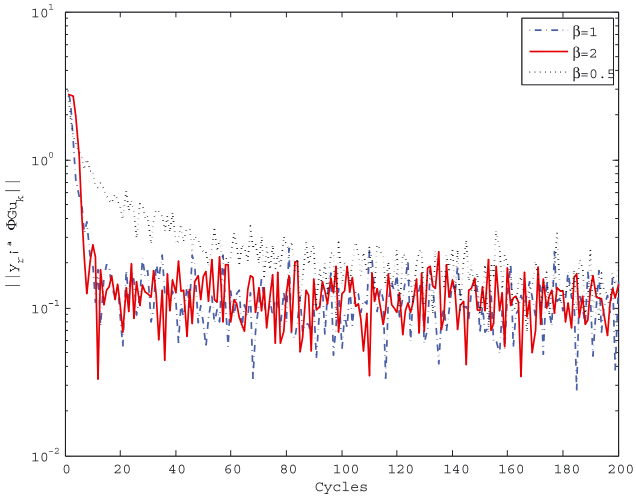

Fig. 2. Norm of the modified tracking error $\|\boldsymbol{y}_r - \Phi \boldsymbol{G}\boldsymbol{u}_k\|$ with different learning gain. [Color figure can be viewed at wileyonlinelibrary.com]

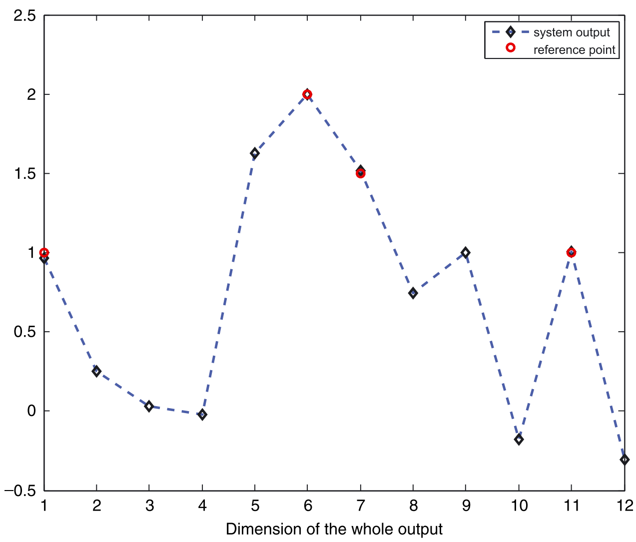

Fig. 3. The last whole output $\boldsymbol{y}_{200}$ and the reference $\boldsymbol{y}_r$. [Color figure can be viewed at wileyonlinelibrary.com]

and $\beta = 0.5$, respectively. That is, the decreasing learning gains are $a_k = 1/k$, $a_k = 2/k$, and $a_k = 0.5/k$, respectively. The modified tracking error profiles along the iteration axis are shown in Fig. 2. One may find little difference in the tracking performance among different learning gain cases as long as the convergence condition is satisfied. Thus different β may mainly affect the design of L.

The whole output of the last iteration $\boldsymbol{y}_{200}$ and the reference points are shown in Fig. 3, where the dashed line with diamonds denotes all components of the output and the four circles denote the desired points. The integers in x-label correspond to $y^{(1)}(1)$, $y^{(2)}(1)$, $y^{(1)}(2)$, $\cdots$, $y^{(2)}(6)$, thus the four circles in Fig. 3 denote $y_d^{(1)}(1)$, $y_d^{(2)}(3)$, $y_d^{(1)}(4)$ and $y_d^{(1)}(6)$. From the figure, one can find that the system output tracks the desired points effectively under the noise environment.

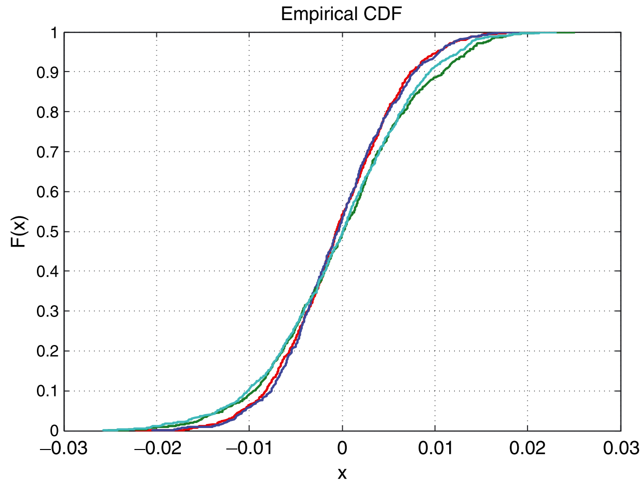

Fig. 4. The empirical CDF for tracking error data at reference points. [Color figure can be viewed at wileyonlinelibrary.com]

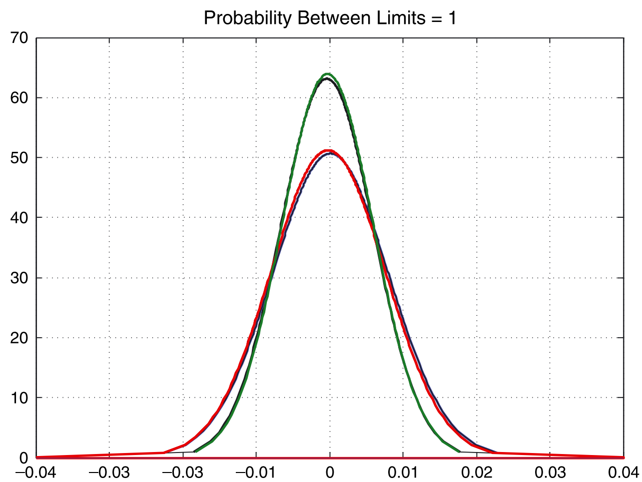

Fig. 5. The probability distribution function based on tracking error data at reference points. [Color figure can be viewed at wileyonlinelibrary.com]

In order to test the asymptotic normality, we repeat the above simulation for 1000 times and collect all the tracking errors at the reference points of the last iteration (the 200th iteration). Fig. 4 shows the empirical cumulative distribution function (CDF) for the tracking error data at each reference point and Fig. 5 plots the probability distribution function of the resulting distribution based on the tracking error data.

V. CONCLUSIONS

An ILC algorithm is proposed for the stochastic point-to-point tracking system in this paper. The general problem formulation is given by introducing a matrix to connect the selected reference points with the original reference. Moreover, a decreasing learning gain is introduced in the conventional P-type update law to suppress stochastic noises. The almost sure convergence of the new algorithm is directly proved in term of modified tracking error. Furthermore, asymptotic normality of the limiting modified tracking error is also provided.

However, in our algorithm, the unrequested outputs are with no consideration, while these outputs leave much freedom to design the algorithm and to seek more optimization objectives if they are observable. For example, whether could the energy of unrequested outputs be minimized simultaneously for saving control efforts? This is an interesting and open problem.

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VI. APPENDIX

The following theorems on stochastic approximation algorithm are cited from [27].

Consider the following recursion with arbitrary initial value η_0,

$$\eta_{k+1} = \eta_k + a_k H_k \eta_k + a_k(\mu_{k+1} + \nu_{k+1}). \tag{15}$$

Theorem 3. Assume the following conditions hold:

A.1 $\{a_k\}$ satisfies that $a_k > 0$, $a_k \xrightarrow[k\to\infty]{} 0$, $\sum_{k=1}^{\infty} a_k = \infty$, and $a_{k+1}^{-1} - a_k^{-1} \to \alpha \geq 0$ as $k \to \infty$;

A.2 $\{\mu_k\}$ and $\{\nu_k\}$ satisfy that $\sum_{k=1}^{\infty} a_k \mu_{k+1} < \infty$ and $\nu_k \to 0$;

A.3 $\{H_k\}$ are $l \times l$ matrices satisfying that $H_k \to H$ and H is stable.

Then $\{\eta_k\}$ generated by (15) tends to zero, *i.e.*, $\eta_k \xrightarrow[k\to\infty]{} 0$.

Theorem 4. Assume A.1 and the following conditions hold

A.2' $\{\mu_k, \mathcal{F}_k\}$ is a martingale difference sequence of l-dimension with $\mathbb{E}(\mu_k|\mathcal{F}_k) = 0$, $\lim_{k\to\infty} \mathbb{E}(\mu_k \mu_k^T|\mathcal{F}_k) = \Gamma$, and $\{\nu_k\}$ satisfies that $\nu_k = o(\sqrt{a_k})$;

A.3' $\{H_k\}$ are $l \times l$ matrices satisfying that $H_k \to H$ and $H + \frac{\alpha}{2}I$ is stable.

Then $\frac{1}{\sqrt{a_k}}\eta_k$, where $\{\eta_k\}$ is generated by (15), is asymptotically normal:

$$\frac{1}{\sqrt{a_k}}\eta_k \xrightarrow[k\to\infty]{d} \mathcal{N}(0, M)$$

where

$$M = \int_0^{\infty} e^{(H+\frac{\alpha}{2}I)z}\Gamma e^{(H^T+\frac{\alpha}{2}I)z} dz.$$

Yun Xu received the B.S. degree in Automation from Beijing Institute of Petrochemical Technology, China, in 2014. Now she is pursuing a M.S. degree at College of Information Science and Technology, Beijing University of Chemical Technology, Beijing, China. Her research interests are in the area of sampled-data iterative learning control and adaptive iterative learning control.

Dong Shen received the B.S. degree in mathematics from Shandong University, Jinan, China, in 2005. He received the Ph.D. degree in mathematics from the Academy of Mathematics and System Science, Chinese Academy of Sciences (CAS), Beijing, China, in 2010. From 2010 to 2012, he was a Post-Doctoral Fellow with the Institute of Automation, CAS. Since 2012, he has been an associate professor with College of Information Science and Technology, Beijing University of Chemical Technology, Beijing, China. His current research interests include iterative learning controls, stochastic control and optimization. He has published more than 40 refereed journal and conference papers. He is the author of *Stochastic Iterative Learning Control* (Science Press, 2016, in Chinese) and co-author of *Iterative Learning Control for Multi-Agent Systems Coordination* (Wiley, 2017). Dr. Shen received IEEE CSS Beijing Chapter Young Author Prize in 2014 and Wentsun Wu Artificial Intelligence Science and Technology Progress Award in 2012.

Xiao-Dong Zhang received the B.S. degree in Automation from Beijing University of Chemical Technology, China, in 2014. Now he is pursuing a M.S. degree at College of Information Science and Technology, Beijing University of Chemical Technology, Beijing, China. His research interests are in the area of soft-sensing and machine learning.