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Convergence analysis for iterative learning control of conformable fractional differential equations

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This paper mainly deals with iterative learning control for the conformable fractional differential equations. The standard P-type, D_α-type, and conformable $PI_\alpha D_\alpha$-type learning updating laws are proposed to derive the convergence results for nonlinear and linear problems varying with the initial state is (not) coincident with the desired initial state. Finally, numerical examples are given to illustrate the results.

KEYWORDS
conformable fractional differential equations, convergence, learning updating laws

1 | INTRODUCTION

As we all know Khalil et al[1] introduced the concept of conformable (local version) fractional derivative, which coincides with the standard (nonlocal version) fractional derivatives[2] on polynomials up to a constant multiple and also can be used to characterize fractional Newton mechanics[3] and the model in mathematical biology.[4] In particular, local version fractional derivative is well behaved and obeys the Leibniz rule and chain rule, which has been proved not well kept for nonlocal version fractional derivative like Riemann-Liouville and Caputo derivatives.[5-7] Meanwhile, it is a natural extension of the usual derivative and can be widely used to establish chain rule, exponential functions, Gronwall's inequality, integration by parts, Taylor power series expansions, Grünwald-Letnikov approach, and calculus of variations for conformable version fractional calculus (see, for example, Abdeljawad[8]). In addition, Laplace transforms,[8] variation of constants methods,[9] and differential transform method[10] are used to find the representation and stability of solutions to linear conformable differential equations, and functional analysis method is used to deal with nonlinear conformable differential equations[11,12] respectively.

Since Uchiyama[13] and Arimoto and Kawamura[14] offered to use the idea of iterative learning strategy to track a desired trajectory, various version iterative updating laws are proposed for different type dynamical systems (see, for example, previous studies[15-21]). Recently, Wang et al[22] proposed conformable D_α-type learning updating law to study iterative learning control for the linear conformable fractional differential equations and established a new convergence result; however, there exists quite a few study on iterative learning control for nonlinear conformable fractional differential equations via P-type and D_α-type iterative updating laws.

In this paper, we consider the iterative learning control for nonlinear conformable fractional differential equations with repetitive properties as follows:

$$\begin{cases} \mathfrak{D}_\alpha^0 x_i(t) = f(t, x_i(t), u_i(t)),\ t \in J := [0, T],\ T > 0, \\ y_i(t) = g(t, x_i(t)) + C(t)u_i(t), \end{cases} \tag{1}$$

and

$$\begin{cases} \mathfrak{D}_\alpha^0 x_i(t) = F(t, x_i(t)) + Q(t)u_i(t),\ t \in J, \\ y_i(t) = P(t)x_i(t), \end{cases} \tag{2}$$

where $\mathfrak{D}_\alpha^0 (0 < \alpha < 1)$ denotes conformable fractional derivative with lower index zero, $x_i(t) \in \mathbb{R}^n, u_i(t) \in \mathbb{R}^r$, $y_i(t) \in \mathbb{R}^m$, $C(t) \in \mathbb{R}^{m\times r}$, $Q(t) \in \mathbb{R}^{n\times r}$, $P(t) \in \mathbb{R}^{m\times n}$, $f : J \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ is continuous, $g : J \times \mathbb{R}^n \to \mathbb{R}^m$ is continuous, and $F : J \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

We apply P-type and D_α-type iterative learning updating laws to study (1) and (2), respectively. In addition, we also apply a conformable $\mathrm{PI}_\alpha D_\alpha$-type iterative learning updating laws to study linear conformable fractional differential equations. The main contribution of this paper is to provide standard analysis technique for iterative learning control for nonlinear and linear conformable fractional differential equations.

The rest of this paper is organized as follows. In Section 2, we collect some spaces, definitions, and lemmas. In Section 3, we give the convergence results of the P-type, D_α-type, and conformable $\mathrm{PI}_\alpha D_\alpha$-type iterative learning control process. Illustrative simulations are given in Section 4 to verify the main results. Section 5 presents the conclusions.

2 | PRELIMINARIES

Let $C(J, \mathbb{R}^n)$ be the space of vector-value continuous functions from $J \to \mathbb{R}^n$. Consider $C(J, \mathbb{R}^n)$ endued with λ-norm $\|x\|_\lambda = \sup_{t\in J}\{e^{-\lambda t}\|x(t)\|\}$ or $\lambda.\alpha$-norm $\|x\|_{\lambda.\alpha} = \sup_{t\in J}\{e^{-\lambda \frac{t^\alpha}{\alpha}}\|x(t)\|\}$ for some $\lambda > 0$ and $0 < \alpha < 1$. For a vector $\omega = (\omega_1, \cdots, \omega_n) \in \mathbb{R}^n$, we consider its vector norm $\|\omega\| = \sqrt{\sum_{i=1}^n \omega_i^2}$ came into being $\|\cdot\|$. For a matrix $M : \mathbb{R}^n \to \mathbb{R}^n$, we consider its matrix norm $\|M\| = \max_{\|x\|=1}\|Mx\|$ came into being from $\|\cdot\|$. Here, vector norms and matrix norms all use symbol $\|\cdot\|$.

Definition 2.1. (see Khalil et al[1], Definition 2.1) The conformable fractional derivative with lower index a of a function $x : [a, \infty) \to \mathbb{R}$ is defined as

$$\mathfrak{D}_\beta^a x(t) = \lim_{\epsilon\to 0} \frac{x(t + \epsilon(t - a)^{1-\beta}) - x(t)}{\epsilon}, \quad t > a, \quad 0 < \beta \le 1,$$
$$\mathfrak{D}_\beta^a x(a) = \lim_{t\to a^+} \mathfrak{D}_\beta^a x(t).$$

Remark 2.2. If $\mathfrak{D}_\beta^a x(t_0)$ exists and is finite, we consider that x is β-differentiable at t_0. If $x : [a, \infty) \to \mathbb{R}$ is a once continuous differential function, then $\mathfrak{D}_\beta^a x(t) = (t - a)^{1-\beta}x'(t)$.

By Remark 2.2, we may conclude that y_d be a β-differentiable expectation function on J provided that $\mathfrak{D}_\beta^a x(t_0)$ exists and is finite or y_d be a once continuous differentiable expectation function on J provided that x' exists.

Theorem 2.3. *(see Pospíšil and Pospíšilová Škripková[12, Lemma 4]) Let $f : J \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ be continuous. A solution $x_i \in C(J, \mathbb{R})$ of (1) with the initial value $x_i(0)$ satisfies the following integral equation:*

$$x_i(t) = x_i(0) + \int_0^t f(\tau, x_i(\tau), u_i(\tau))\tau^{\alpha-1}d\tau. \tag{3}$$

Let y_d be a once continuous differentiable expectation function on J and u_d be a desired control variable. By Theorem 2.3, if (1) with initial value $x_d(0)$, then $y_d(t) = g(t, x_d(t)) + C(t)u_d(t)$, where

$$x_d(t) = x_d(0) + \int_0^t f(\tau, x_d(\tau), u_d(\tau))\tau^{\alpha-1}d\tau.$$

By Pospíšil and Pospíšilová Škripková[12, Lemma 4] again, if (2) with initial value $x_d(0)$, then $y_d(t) = P(t)x_d(t)$, where

$$x_d(t) = x_d(0) + \int_0^t (F(\tau, x_d(\tau)) + Q(\tau)u_d(\tau))\tau^{\alpha-1}d\tau.$$

For simplification, we set

$$\delta u_i(t) = u_d(t) - u_i(t), \quad \delta x_i(t) = x_d(t) - x_i(t), \quad e_i(t) = y_d(t) - y_i(t). \tag{4}$$

Lemma 2.4. *(see Ortega and Rheinboldt[23, 2.2.8, Chapter 2]) Let $A \in L(\mathbb{R}^n)$. For a given $\varepsilon > 0$,*

$$\|A\| \le \rho(A) + \varepsilon,$$

where $\rho(A)$ denotes spectral radius of matrix A.

Lemma 2.5. *(see Bainov and Simeonov[24], Theorem 1) Let $u(t)$ and $b(t)$ be nonnegative continuous functions for $t \ge \alpha$, and let*

$$u(t) \le a + \int_\alpha^t b(s)u(s)ds, \ t \ge \alpha,$$

where $\alpha \ge 0$ is a constant. Then

$$u(t) \le a e^{\int_\alpha^t b(s)ds}, \ t \ge \alpha.$$

3 | CONVERGENCE RESULTS

3.1 | P-type learning law

We introduce the following assumptions:

(A_1) $f : J \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ is continuous, and there exists a $L_f > 0$ such that

$$\|f(t, x_1, u_1) - f(t, x_2, u_2)\| \le L_f(\|x_1 - x_2\| + \|u_1 - u_2\|),$$

where $\forall t \in J, \ \forall x_1, x_2 \in \mathbb{R}^n, \ \forall u_1, u_2 \in \mathbb{R}^r$.

(A_2) $g : J \times \mathbb{R}^n \to \mathbb{R}^m$ is continuous, and there exists a $H > 0$ such that

$$\|g(t, x_1) - g(t, x_2)\| \le H\|x_1 - x_2\|,$$

where $\forall t \in [0, T], \ x_1, x_2 \in \mathbb{R}^n$.

(A_3) The existence of u_d and x_d makes the output of the system (1) to be y_d.

Consider the following open-loop P-type learning law:

$$u_{i+1}(t) = u_i(t) + \Gamma(t)e_i(t), \tag{5}$$

where the subscript i denotes the iteration and $\Gamma(t)$ denotes the learning gain.

Theorem 3.1. *Consider (1) associated with (A_1), (A_2), (A_3) and $x_i(0) = x_d(0)(i = 1, 2, \dots)$. If (5) is applied to (1), then the output $y_i(t)$ uniformly converges to $y_d(t)$ provided that $\alpha > \frac{1}{p}, p > 1$ and*

$$\sup_{t \in J} \|I - \Gamma(t)C(t)\| < 1. \tag{6}$$

Proof. By (5) and (4), we get

$$\begin{aligned}\delta u_{i+1}(t) &= \delta u_i(t) - \Gamma(t)e_i(t)\\ &= \delta u_i(t) - \Gamma(t)(g(t,x_d(t)) - g(t,x_i(t)) + C(t)\delta u_i(t))\\ &= [I - \Gamma(t)C(t)]\delta u_i(t) - \Gamma(t)(g(t,x_d(t)) - g(t,x_i(t))),\end{aligned}$$

where I denotes identity matrix.

By (A_2) and we taking the matrix norm for the above equality, we have

$$\begin{aligned}\|\delta u_{i+1}(t)\| &\le \|I - \Gamma(t)C(t)\|\|\delta u_i(t)\| + \|\Gamma(t)\|\|g(t,x_d(t)) - g(t,x_i(t))\|\\ &\le \|I - \Gamma(t)C(t)\|\|\delta u_i(t)\| + H\|\Gamma(t)\|\|\delta x_i(t)\|.\end{aligned}$$

Multiplying both the sides by $e^{-\lambda t}$, one has

$$\begin{aligned}e^{-\lambda t}\|\delta u_{i+1}(t)\| &\le \|I - \Gamma(t)C(t)\|e^{-\lambda t}\|\delta u_i(t)\| + H\|\Gamma(t)\|e^{-\lambda t}\|\delta x_i(t)\|\\ &\le \|I - \Gamma(t)C(t)\|e^{-\lambda t}\|\delta u_i(t)\| + n_1 e^{-\lambda t}\|\delta x_i(t)\|,\end{aligned} \tag{7}$$

where $n_1 = \sup_{t\in J} H\|\Gamma(t)\|$. Now, taking λ-norm of (7), we get

$$\|\delta u_{i+1}\|_\lambda \le \sup_{t\in[0,T]} \|I - \Gamma(t)C(t)\|\|\delta u_i\|_\lambda + n_1\|\delta x_i\|_\lambda. \tag{8}$$

According to (A_1) and (3), we can get the following inequality by taking the matrix norm:

$$\begin{aligned}\|\delta x_i(t)\| &\le \int_0^t (\|f(\tau,x_d(\tau),u_d(\tau)) - f(\tau,x_i(\tau),u_i(\tau))\|)\tau^{\alpha-1}d\tau\\ &\le L_f\int_0^t \|\delta x_i(\tau)\|\tau^{\alpha-1}d\tau + L_f\int_0^t \|\delta u_i(\tau)\|\tau^{\alpha-1}d\tau.\end{aligned}$$

By using Lemma 2.5, one can get

$$\begin{aligned}\|\delta x_i(t)\| &\le L_f\int_0^t \|\delta u_i(\tau)\|\tau^{\alpha-1}d\tau e^{\int_0^t L_f\tau^{\alpha-1}d\tau}\\ &\le L_f\int_0^t e^{\lambda\tau}\tau^{\alpha-1}d\tau\|\delta u_i\|_\lambda e^{L_f\frac{t^\alpha}{\alpha}}.\end{aligned} \tag{9}$$

For any given $\alpha > 0$, the existence of $p > 1$ makes $\alpha > \frac{1}{p}$. Then we can see the $\exists q > 1$ makes $\frac{1}{p} + \frac{1}{q} = 1$. By applying Hölder inequality,

$$\begin{aligned}\int_0^t e^{\lambda\tau}\tau^{\alpha-1}d\tau &\le \left(\int_0^t e^{\lambda\tau p}d\tau\right)^{\frac{1}{p}}\left(\int_0^t \tau^{(\alpha-1)\times q}d\tau\right)^{\frac{1}{q}}\\ &\le \frac{e^{\lambda t}}{\sqrt[p]{p}\sqrt[p]{\lambda}}\left(\frac{t^{q\alpha-q+1}}{q\alpha-q+1}\right)^{\frac{1}{q}}\\ &\le \frac{1}{\lambda}e^{\lambda t}\frac{T^{\alpha-\frac{1}{p}}}{\sqrt[q]{q\alpha-q+1}},\ (\lambda \ge 1).\end{aligned} \tag{10}$$

Substituting (10) into (9), we have

$$\|\delta x_i(t)\| \le L_f\frac{1}{\lambda}e^{\lambda t}\frac{T^{\alpha-\frac{1}{p}}}{\sqrt[q]{q\alpha-q+1}}\|\delta u_i\|_\lambda e^{L_f\frac{T^\alpha}{\alpha}}.$$

Then taking λ-norm, one can get

$$\|\delta x_i\|_\lambda \le L_f\frac{T^{\alpha-\frac{1}{p}}}{\lambda\sqrt[q]{q\alpha-q+1}}\|\delta u_i\|_\lambda e^{L_f\frac{T^\alpha}{\alpha}}. \tag{11}$$

Because of (8) and (11), we can have

$$\begin{aligned}\|\delta u_{i+1}\|_\lambda &\le \sup_{t\in J}\|I-\Gamma(t)C(t)\|\|\delta u_i\|_\lambda + n_1 L_f \frac{T^{\alpha-\frac{1}{p}}}{\lambda\sqrt[q]{q\alpha-q+1}}\|\delta u_i\|_\lambda e^{L_f\frac{T^\alpha}{\alpha}}\\ &\le \sup_{t\in J]}\|I-\Gamma(t)C(t)\|\|\delta u_i\|_\lambda + \frac{n_2}{\lambda}\|\delta u_i\|_\lambda,\end{aligned}$$

where $n_2 = \max\left\{ n_1 e^{L_f\frac{\tau^\alpha}{\alpha}}, n_1 L_f \frac{T^{\alpha-\frac{1}{p}}}{\sqrt[q]{q\alpha-q+1}} e^{L_f\frac{T^\alpha}{\alpha}}\right\}$. By (6), we can get $\|\delta u_i\|_\lambda \to 0$ for some λ large enough.

Note that

$$e_i(t) = g(t, x_d(t)) - g(t, x_i(t)) + C(t)u_d(t) - C(t)u_i(t)$$

and

$$\|e_i(t)\| \le H\|\delta x_i(t)\| + \|C(t)\|\|\delta u_i(t)\|. \tag{12}$$

Substituting (11) into (12) and taking the λ-norm, we get

$$\begin{aligned}\|e_i\|_\lambda &\le H\|\delta x_i\|_\lambda + \sup_{t\in J}\|C(t)\|\|\delta u_i\|_\lambda\\ &\le HL_f\frac{T^{\alpha-\frac{1}{p}}}{\lambda\sqrt[q]{q\alpha-q+1}}\|\delta u_i\|_\lambda e^{L_f\frac{T^\alpha}{\alpha}} + \sup_{t\in J}\|C(t)\|\|\delta u_i\|_\lambda.\end{aligned}$$

This implies that $\|e_i\|_\lambda \to 0$ due to $\|\delta u_i\|_\lambda \to 0$ when λ is big enough. So $y_i(t)$ is convergence uniformly to $y_d(t)$ in J. The proof is completed. □

3.2 | D_α-type learning law

We introduce the following assumptions:

(B_1) $F : J\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous, and there exists a $L_F > 0$ such that

$$\|F(t,x_1) - F(t,x_2)\| \le L_F\|x_1 - x_2\|, \tag{13}$$

where $\forall t\in J,\ \forall x_1, x_2 \in \mathbb{R}^n$.

(B_2) The existence of u_d and x_d makes the output of the system (2) to be y_d.

$(B_3) P(t) \in \mathbb{R}^{m\times n}$, $Q(t) \in \mathbb{R}^{n\times r}$, $Q(t), P(t)$, and $\dot{P}(t)$ are uniformly bounded for all $t \in J$, where $P(t) = (p_1(t), p_2(t), \dots, p_n(t))^T$ and $\dot{P}(t) = \left(\frac{d}{dt}p_1(t), \frac{d}{dt}p_2(t), \dots, \frac{d}{dt}p_n(t)\right)^T$.

Consider the following open-loop D_α-type learning law:

$$u_{i+1}(t) = u_i(t) + \Gamma(t)\mathfrak{D}_\alpha^0 e_i(t), \tag{14}$$

where the subscript i denotes the iteration and $\Gamma(t)$ denotes the learning gain.

Theorem 3.2. *Consider (2) associated with (B_1), (B_2), (B_3) and $x_i(0) = x_d(0) (i = 1, 2, \dots)$. If (14) is applied to (2), then the output $y_i(t)$ uniformly converges to $y_d(t)$ provided that $\alpha > \frac{1}{p}, p > 1$ and*

$$\sup_{t\in J}\|(I - \Gamma(t)P(t)Q(t))\| < 1. \tag{15}$$

Proof. From (2) and (14), one has

$$\begin{aligned}\delta u_{i+1}(t) &= \delta u_i(t) - \Gamma(t)\mathfrak{D}_\alpha^0 e_i(t)\\ &= \delta u_i(t) - \Gamma(t)\mathfrak{D}_\alpha^0(P(t)\delta x_i(t))\\ &= \delta u_i(t) - \Gamma(t)\mathfrak{D}_\alpha^0 P(t)\delta x_i(t) - \Gamma(t)P(t)\mathfrak{D}_\alpha^0\delta x_i(t)\\ &= \delta u_i(t) - \Gamma(t)\mathfrak{D}_\alpha^0 P(t)\delta x_i(t) - \Gamma(t)P(t)(F(t,x_d(t)) - F(t,x_i(t)) + Q(t)\delta u_i(t))\\ &= [I - \Gamma(t)P(t)Q(t)]\delta u_i(t) - \Gamma(t)\dot{P}(t)t^{1-\alpha}\delta x_i(t) - \Gamma(t)P(t)(F(t,x_d(t)) - F(t,x_i(t))).\end{aligned}$$

Using (B_1) and (B_3) and taking the matrix norm for both the sides of the above equality,

$$\begin{aligned}\|\delta u_{i+1}(t)\| &\leq \|I-\Gamma(t)P(t)Q(t)\|\|\delta u_i(t)\| + \|\Gamma(t)\dot{P}(t)t^{1-\alpha}\|\|\delta x_i(t)\| \\ &\quad + \|\Gamma(t)P(t)\|\|F(t,x_d(t))-F(t,x_i(t))\| \\ &\leq \|I-\Gamma(t)P(t)Q(t)\|\|\delta u_i(t)\| + \|\Gamma(t)\dot{P}(t)t^{1-\alpha}\|\|\delta x_i(t)\| + \|\Gamma(t)P(t)\|H\|\delta x_i(t)\| \\ &\leq \|I-\Gamma(t)P(t)Q(t)\|\|\delta u_i(t)\| + (\|\Gamma(t)\|\|\dot{P}(t)\|t^{1-\alpha} + L_F\|\Gamma(t)\|\|P(t)\|)\|\delta x_i(t)\|.\end{aligned} \tag{16}$$

Multiplying both the sides of (16) by $e^{-\lambda t}$, we get

$$\begin{aligned}e^{-\lambda t}\|\delta u_{i+1}(t)\| &\leq \|I-\Gamma(t)P(t)Q(t)\|e^{-\lambda t}\|\delta u_i(t)\| + (\|\Gamma(t)\dot{P}(t)t^{1-\alpha}\| + L_F\|\Gamma(t)P(t)\|)e^{-\lambda t}\|\delta x_i(t)\| \\ &\leq \|I-\Gamma(t)P(t)Q(t)\|e^{-\lambda t}\|\delta u_i(t)\| + b_1 e^{-\lambda t}\|\delta x_i(t)\|,\end{aligned} \tag{17}$$

where $b_1 = \sup_{t\in J}\{\|\Gamma(t)\|\|\dot{P}(t)\|t^{1-\alpha} + L_F\|\Gamma(t)\|\|P(t)\|\}$.

Now, taking λ-norm of (17), one can get

$$\|\delta u_{i+1}\|_\lambda \leq \sup_{t\in J}\|I-\Gamma(t)P(t)Q(t)\|\|\delta u_i(t)\|_\lambda + b_1\|\delta x_i(t)\|_\lambda. \tag{18}$$

According to (B_1) and (B_3), we can obtain the following inequality by taking the matrix norm:

$$\begin{aligned}\|\delta x_i(t)\| &\leq \int_0^t (\|F(\tau,x_d(\tau))-F(\tau,x_i(\tau))\| + \|Q(\tau)\delta u_i(\tau)\|)\tau^{\alpha-1}d\tau \\ &\leq L_F\int_0^t \|\delta x_i(\tau)\|\tau^{\alpha-1}d\tau + \int_0^t \|Q(\tau)\delta u_i(\tau)\|\tau^{\alpha-1}d\tau.\end{aligned}$$

Using Lemma 2.5, we have

$$\begin{aligned}\|\delta x_i(t)\| &\leq \int_0^t \|Q(\tau)\delta u_i(\tau)\|\tau^{\alpha-1}d\tau e^{\int_0^t L_F\tau^{\alpha-1}d\tau} \\ &\leq \int_0^t \|Q(\tau)\delta u_i(\tau)\|\tau^{\alpha-1}d\tau e^{L_F\frac{t^\alpha}{\alpha}} \\ &\leq \int_0^t e^{\lambda\tau}\tau^{\alpha-1}d\tau\|Q(t)\|\|\delta u_i\|_\lambda e^{L_F\frac{t^\alpha}{\alpha}}.\end{aligned} \tag{19}$$

Substituting (10) into (19), one can get

$$\|\delta x_i(t)\| \leq \frac{1}{\lambda}e^{\lambda t}\frac{T^{\alpha-\frac{1}{p}}}{\sqrt[q]{q\alpha-q+1}}\|Q(t)\|\|\delta u_i\|_\lambda e^{L_F\frac{T^\alpha}{\alpha}}.$$

Then taking λ-norm, we have

$$\|\delta x_i(t)\|_\lambda \leq \frac{1}{\lambda}\frac{T^{\alpha-\frac{1}{p}}}{\sqrt[q]{q\alpha-q+1}}\|Q(t)\|\|\delta u_i\|_\lambda e^{L_F\frac{T^\alpha}{\alpha}}. \tag{20}$$

Because of (18) and (20), we get

$$\begin{aligned}\|\delta u_{i+1}\|_\lambda &\leq \sup_{t\in J}\|I-\Gamma(t)P(t)Q(t)\|\|\delta u_i\|_\lambda + b_1\frac{1}{\lambda}\frac{T^{\alpha-\frac{1}{p}}}{\sqrt[q]{q\alpha-q+1}}\|Q(t)\|\|\delta u_i\|_\lambda e^{L_F\frac{T^\alpha}{\alpha}} \\ &\leq \sup_{t\in J}\|I-\Gamma(t)P(t)Q(t)\|\|\delta u_i\|_\lambda + \frac{b_2}{\lambda}\|\delta u_i\|_\lambda,\end{aligned}$$

where $b_2 = \max\left\{ b_1 e^{L_F \frac{T^\alpha}{\alpha}}, \sup_{t\in J} \frac{T^{\alpha-\frac{1}{p}}}{\sqrt[q]{q\alpha-q+1}} \|Q(t)\| e^{L_F \frac{T^\alpha}{\alpha}} \right\}$. By (15), we can obtain $\|\delta u_i\|_\lambda \to 0$ for some λ large enough.

Note that

$$e_i(t) = y_d(t) - y_i(t) = P(t)\delta x_i(t)$$

and

$$\|e_i(t)\| \leq \|P(t)\|\|\delta x_i(t)\|. \tag{21}$$

Substituting (20) into (21) and taking the λ-norm, we have

$$\begin{aligned} \|e_i\|_\lambda &\leq \|P(t)\|\|\delta x_i(t)\|_\lambda \\ &\leq \|P(t)\| \frac{b_2}{\lambda} \|\delta u_i\|_\lambda. \end{aligned}$$

This shows that $\|e_i\|_\lambda \to 0$. So the proof is completed. □

3.3 | Conformable $\text{PI}_\alpha D_\alpha$-type learning law

In this section, we apply $\text{PI}_\alpha D_\alpha$-type learning law to study the convergence of the following linear system

$$\begin{cases} \mathfrak{D}_\alpha^0 x_i(t) = Mx_i(t) + Qu_i(t), t \in J. \\ y_i(t) = Px_i(t), \end{cases} \tag{22}$$

where $x_i(t) \in R^n, u_i(t) \in R^r, y_i(t) \in R^m, M \in \mathbb{R}^{n\times n}, Q \in \mathbb{R}^{n\times r}, P \in \mathbb{R}^{m\times n}$. Obviously, a state $x_i(t)$ of (22) with the initial state $x_i(0)$ and control function u_i has the form (see Li et al[9, Theorem 3.3]):

$$x_i(t) = x_i(0)\Phi(t,0) + \int_0^t \Phi(t,\tau)Qu_i(\tau)\tau^{\alpha-1}d\tau,$$

where $\Phi(t,\tau) := e^{M\frac{t^\alpha}{\alpha}} e^{-M\frac{\tau^\alpha}{\alpha}}$ is called state transition matrix determined by M in (22) satisfying

$$\begin{aligned} \mathfrak{D}_\alpha^0 \Phi(t,\tau) &= M\Phi(t,\tau)t^{1-\alpha}, \\ \Phi(t_0,t_0) &= I,\ \ t_0 \in J. \end{aligned}$$

For (22), we give a reachable desired trajectory $y_d(t)(t \in J)$ and use a conformable $\text{PI}_\alpha D_\alpha$-type learning law as

$$u_{i+1}(t) = u_i(t) + \Gamma \mathfrak{D}_\alpha^0 e_i(t) + Le_i(t) + \psi \int_0^t e_i(\tau)\tau^{\alpha-1}d\tau, \tag{23}$$

where the subscript i denotes the iteration and Γ, L, and ψ denote the learning gain.

Theorem 3.3. *Applying (23) to (22) associated with $x_i(0) = x^0 (i = 0,1,2,\ldots), y_0(0) = y_d(0)$. Then the output $y_i(t)$ uniformly converges to $y_d(t)$ provided that*

$$\rho(I - PQ\Gamma) \leq \bar{\rho} < 1. \tag{24}$$

Proof. By (22) and $x_i(0) = x^0 (i = 0,1,2,\ldots), y_0(0) = y_d(0)$, we have

$$y_{i+1}(0) = Px_{i+1}(0) = Px_i(0) = y_i(0).$$

Hence, we get $e_i(0) = 0 (i = 0,1,2,\ldots)$.

The output error at the ith iteration can be written as

$$\begin{aligned} e_{i+1}(t) &= e_i(t) - \int_0^t P\Phi(t,\tau)Q(u_{i+1}(\tau) - u_i(\tau))\tau^{\alpha-1}d\tau \\ &= e_i(t) - \int_0^t P\Phi(t,\tau)Q\left[\Gamma\mathfrak{D}_\alpha^0 e_i(\tau) + Le_i(\tau) + \psi\int_0^\tau e_i(\sigma)\sigma^{\alpha-1}d\sigma\right]\tau^{\alpha-1}d\tau. \end{aligned} \tag{25}$$

Set $R(t,\tau) = P\Phi(t,\tau)Q\Gamma$. From the method of integration by parts, we obtain

$$\begin{aligned} &\int_0^t R(t,\tau)\mathfrak{D}_\alpha^0 e_i(\tau)\tau^{\alpha-1}d\tau \\ &= \int_0^t R(t,\tau)\dot{e}_i(\tau)\tau^{1-\alpha}\tau^{\alpha-1}d\tau \\ &= \int_0^t R(t,\tau)\dot{e}_i(\tau)d\tau \\ &= R(t,\tau)e_i(\tau)|_0^t - \int_0^t \frac{\partial}{\partial\tau}R(t,\tau)e_i(\tau)d\tau \\ &= PQ\Gamma e_i(t) - \int_0^t \frac{\partial}{\partial\tau}R(t,\tau)e_i(\tau)d\tau. \end{aligned} \tag{26}$$

Replace (26) into (25), one can get

$$\begin{aligned} e_{i+1}(t) = [I - PQ\Gamma]e_i(t) &+ \int_0^t \frac{\partial}{\partial\tau}R(t,\tau)e_i(\tau)d\tau \\ &- \int_0^t P\Phi(t,\tau)QLe_i(\tau)\tau^{\alpha-1}d\tau \\ &- \int_0^t\int_0^\tau P\Phi(t,\tau)Q\psi e_i(\sigma)\sigma^{\alpha-1}\tau^{\alpha-1}d\sigma d\tau. \end{aligned} \tag{27}$$

According to Lemma 2.4, for each fixed t and the given $\varepsilon > 0$, there exists a nonsingular matrix $K \in \mathbb{R}^{m\times m}$ satisfying

$$\|K[I - PQ\Gamma]K^{-1}\| \le \rho(I - PQ\Gamma) + \varepsilon < \bar{\rho} + \varepsilon.$$

Then multiplied both the sides of (27) by K and take stand norm, we have

$$\begin{aligned} \|Ke_{i+1}(t)\| \le\ & \|K[I - PQ\Gamma]K^{-1}\|\|Ke_i(t)\| \\ &+ \int_0^t \|K\frac{\partial}{\partial\tau}R(t,\tau)K^{-1}\|\|Ke_i(\tau)\|d\tau \\ &+ \int_0^t \|KP\Phi(t,\tau)QLK^{-1}\|\|Ke_i(\tau)\|\tau^{\alpha-1}d\tau \\ &+ \int_0^t\int_0^\tau \|KP\Phi(t,\tau)Q\psi(\tau)K^{-1}\|\|Ke_i(\sigma)\|\sigma^{\alpha-1}\tau^{\alpha-1}d\sigma d\tau \\ \le\ & (\rho(I - PQ\Gamma) + \varepsilon)\|Ke_i(t)\| + \int_0^t 2c_1\|Ke_i(\tau)\|\tau^{\alpha-1}d\tau \\ &+ \int_0^t\int_0^\tau c_2\|Ke_i(\sigma)\|\sigma^{\alpha-1}\tau^{\alpha-1}d\sigma d\tau, \end{aligned}$$

where $c_1 = \max\{\sup_{t,\tau\in J}\|K\frac{\partial}{\partial\tau}R(t,\tau)K^{-1}\|, \sup_{t,\tau\in J}\|KP\Phi(t,\tau)QLK^{-1}\|\}$ and $c_2 = \sup_{t,\tau\in J}\|KP\Phi(t,\tau)Q\psi(\tau)K^{-1}\|$.

Then multiplying both the sides by $e^{-\lambda\frac{t^\alpha}{\alpha}}(\lambda > 0)$, one can obtain

$$\begin{aligned}
\|Ke_{i+1}\|_{\lambda.\alpha} &\le (\bar{\rho}+\epsilon)\|Ke_i\|_{\lambda.\alpha} + 2c_1\int_0^t e^{-\lambda\left(\frac{t^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)}\tau^{\alpha-1}\|Ke_i\|_{\lambda.\alpha}d\tau \\
&\quad + c_2\int_0^t\int_0^\tau e^{-\lambda\left(\frac{t^\alpha}{\alpha}-\frac{\sigma^\alpha}{\alpha}\right)}\sigma^{\alpha-1}\tau^{\alpha-1}\|Ke_i\|_{\lambda.\alpha}d\sigma d\tau \\
&\le [\bar{\rho}+\epsilon+2c_1\int_0^{\frac{t^\alpha}{\alpha}} e^{-\lambda\left(\frac{t^\alpha}{\alpha}-\frac{\tau^\alpha}{\alpha}\right)}d\frac{\tau^\alpha}{\alpha} \\
&\quad + c_2\int_0^{\frac{t^\alpha}{\alpha}}\int_0^{\frac{\tau^\alpha}{\alpha}} e^{-\lambda\left(\frac{t^\alpha}{\alpha}-\frac{\sigma^\alpha}{\alpha}\right)}d\frac{\sigma^\alpha}{\alpha}d\frac{\tau^\alpha}{\alpha}]\|Ke_i\|_{\lambda.\alpha} \\
&\le \left[\bar{\rho}+\epsilon+2c_1\frac{1-e^{-\lambda\frac{t^\alpha}{\alpha}}}{\lambda} + c_2 e^{-\lambda\frac{t^\alpha}{\alpha}}\int_0^{\frac{t^\alpha}{\alpha}}\frac{e^{\lambda\frac{\tau^\alpha}{\alpha}}-1}{\lambda}d\frac{\tau^\alpha}{\alpha}\right]\|Ke_i\|_{\lambda.\alpha} \\
&\le \left[\bar{\rho}+\epsilon+2c_1\frac{1-e^{-\lambda\frac{t^\alpha}{\alpha}}}{\lambda} + c_2\frac{1-e^{-\lambda\frac{t^\alpha}{\alpha}}-\lambda\frac{t^\alpha}{\alpha}e^{-\lambda\frac{t^\alpha}{\alpha}}}{\lambda^2}\right]\|Ke_i\|_{\lambda.\alpha} \\
&\le \tilde{\rho}\|Ke_i\|_{\lambda.\alpha},
\end{aligned} \tag{28}$$

where $\tilde{\rho} = \bar{\rho}+\epsilon+2c_1\frac{1}{\lambda}+c_2\frac{1}{\lambda^2}$. By the arbitrariness of ϵ via (24), one can derive $\tilde{\rho} < 1$ for some λ sufficiently enough. Therefore, $\lim_{k\to\infty}\|Ke_i\|_{\lambda.\alpha} = 0$. Finally, note that

$$\|e_i\|_{\lambda.\alpha} \le \|K^{-1}\|\|Ke_i\|_{\lambda.\alpha},$$

the proof is completed. □

Remark 3.4. In this theorem, the initial state $x_k(0)$ is reset to arbitrary value x^0, which is not necessarily to the desired initial state $x_d(0)$. This is a distinct difference between the theorem and the previous two theorems.

4 | NUMERICAL EXAMPLES

Example 4.1. Let $n = r = m = 2$. Consider the following nonlinear control system:

$$\begin{cases} \mathfrak{D}_\alpha^0 x(t) = f(t, x(t), u(t)), \\ y(t) = g(t, x(t)) + C(t)u(t), \ t \in [0,1], \end{cases}$$

where

$$f(t,x(t),u(t)) = \begin{pmatrix} 3\sin(u(t)) + 2\sin(x(t)) \\ 2\sin(u(t)) + 3\sin(x(t)) \end{pmatrix},$$

$$y(t) = g(t,x(t)) + C(t)u(t) = \begin{pmatrix} 2x(t) + \sin(x(t)) + 0.1u(t) \\ x(t) + 2\sin(x(t)) + 0.1u(t) \end{pmatrix},$$

and

$$C(t) = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix},$$

$$\alpha = 0.9.$$

Set $x_i(0) = x_d(0)(i = 1, 2, \cdots)$.

Consider P-type learning law as

$$u_{i+1}(t) = u_i(t) + \Gamma(t)e_i(t), \ t \in [0,1],$$

where we choose

$$\Gamma(t) = \begin{pmatrix} 10 & 2 \\ 0 & 10 \end{pmatrix},$$

and given desired reference trajectory as

$$y_d(t) = \begin{pmatrix} 12t(1-t) \\ 15 \sin t \end{pmatrix}, \quad t \in [0, 1],$$

where we mark

$$u_i(t) = \begin{pmatrix} u_{1,i}(t) \\ u_{2,i}(t) \end{pmatrix}, \quad y_d(t) = \begin{pmatrix} y_{1,d}(t) \\ y_{2,d}(t) \end{pmatrix}, \quad e_i(t) = \begin{pmatrix} e_{1,i}(t) \\ e_{2,i}(t) \end{pmatrix}.$$

It is obviously that the state $x_i \in \mathbb{R}^2$, output $y_i \in \mathbb{R}^2$, and input $u_i \in \mathbb{R}^2$. One can see that $\sup_{t\in[0,1]} \|I - \Gamma(t)C(t)\| = 0.2 < 1$. Obviously, the output $y_i(t)$ can track the reference trajectory $y_d(t)$ effectively. In specific applications, when the accuracy of the error reaches the requirements, we can stop iterating. Thus, Theorem 3.1 can be applied to Example 4.1.

Figure 1 shows the tracking performance of Example 4.1. The blue line represents the output track of 1st, 22th, 43th, 64th, and 85th runs, and the red star represents reference trajectory.

Figure 2 shows the numerical simulation results, and the 100th iteration error is 2.26×10^{-11}.

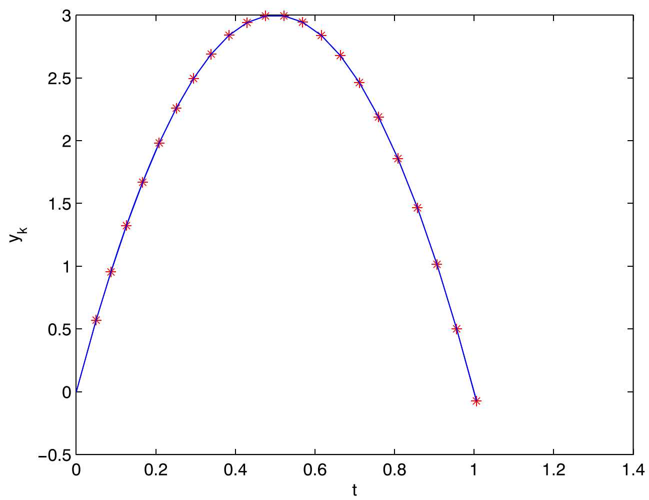

FIGURE 1 The tracking performance of Example 4.1 [Colour figure can be viewed at wileyonlinelibrary.com]

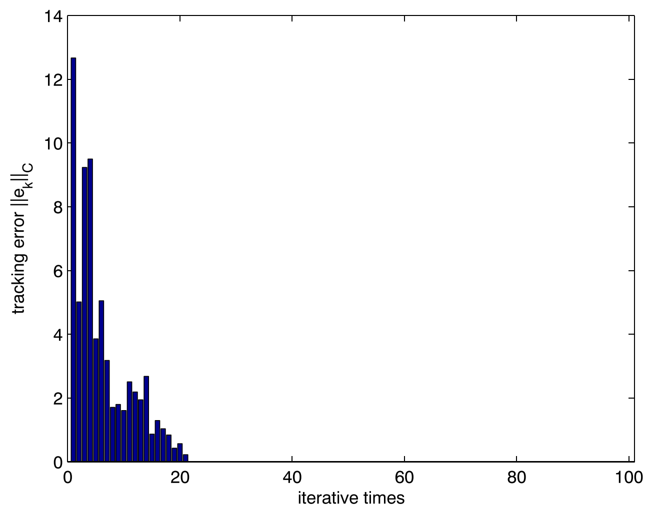

FIGURE 2 The the numerical simulation results of Example 4.1 [Colour figure can be viewed at wileyonlinelibrary.com]

Example 4.2. Let $n = r = m = 2$. Consider the following nonlinear control system:

$$\begin{cases} \mathfrak{D}^0_\alpha x(t) = F(t, x(t)) + Q(t)u(t), \\ y(t) = P(t)x(t), \ t \in [0, 1], \end{cases} \tag{29}$$

where

$$F(t, x(t)) = \begin{pmatrix} 0.5x(t) + \sin(x(t)) \\ 0.5x(t) + 0.5\sin(x(t)) \end{pmatrix},$$
$$Q(t) = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}, \quad P(t) = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix},$$
$$\alpha = 0.9.$$

Set $x_i(0) = x_d(0)(i = 1, 2, \dots)$.

The conformable D_α-type learning law is known as

$$u_{i+1}(t) = u_i(t) + \Gamma(t)\mathfrak{D}^0_\alpha e_i(t), \ t \in [0, 1],$$

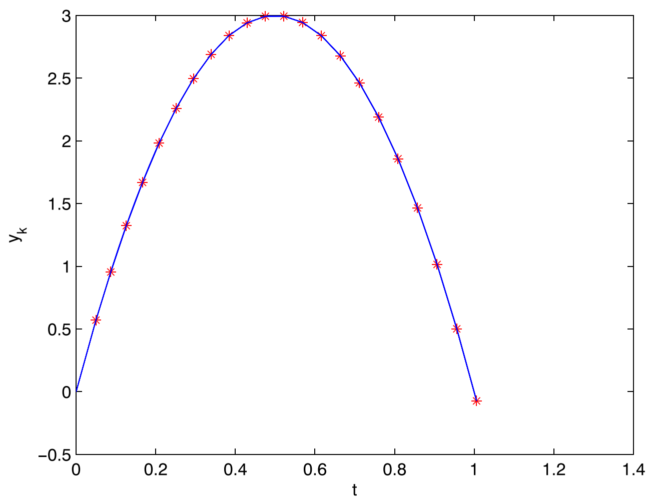

FIGURE 3 The tracking performance of Example 4.2 [Colour figure can be viewed at wileyonlinelibrary.com]

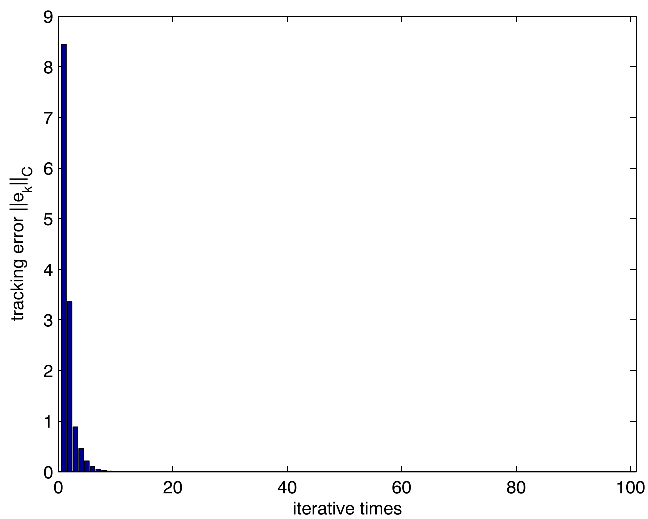

FIGURE 4 The the numerical simulation results of Example 4.2 [Colour figure can be viewed at wileyonlinelibrary.com]

where we choose

$$\Gamma(t) = \begin{pmatrix} 10 & 2 \\ 0 & 10 \end{pmatrix},$$

and given desired reference trajectory as

$$y_d(t) = \begin{pmatrix} 12t(1-t) \\ 10\sin t \end{pmatrix}, t \in [0,1],$$

where we mark

$$u_i(t) = \begin{pmatrix} u_{1,i}(t) \\ u_{2,i}(t) \end{pmatrix}, y_d(t) = \begin{pmatrix} y_{1,d}(t) \\ y_{2,d}(t) \end{pmatrix}, e_i(t) = \begin{pmatrix} e_{1,i}(t) \\ e_{2,i}(t) \end{pmatrix}.$$

It is obviously that the state $x_i \in \mathbb{R}^2$, output $y_i \in \mathbb{R}^2$, and input $u_i \in \mathbb{R}^2$. One can see that $\sup_{t\in[0,1]} \|I - \Gamma(t)P(t)Q(t)\| = 0.54 < 1$. Obviously, the output $y_i(t)$ can track the reference trajectory $y_d(t)$ effectively. In specific applications, when the accuracy of the error reaches the requirements, we can stop iterating. Thus, Theorem 3.2 can be applied to Example 4.2.

Figure 3 shows the tracking performance of Example 4.2. The blue line represents the output track of 1st, 10th, 20th, … , 90th, and 100th runs, and the red star represents reference trajectory.

Figure 4 shows the numerical simulation results and the 100th iteration error is 8.88×10^{-16}.

Example 4.3. Let $n = r = m = 2$. Consider the following linear control system:

$$\begin{cases} \mathfrak{D}_\alpha^0 x(t) = Mx(t) + Qu(t), \\ y(t) = Px(t), t \in [0,1], \end{cases} \tag{30}$$

where

$$M = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \quad Q = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0.5 \\ 0 & 1 \end{pmatrix},$$
$$\alpha = 0.9.$$

Set $x_i(0) = x^0 (i = 0, 1, 2, \dots), y_0(0) = y_d(0)$.

The conformable $\mathrm{PI}_\alpha D_\alpha$-type learning law is known as

$$u_{i+1}(t) = u_i(t) + \Gamma \mathfrak{D}_\alpha^0 e_i(t) + Le_i(t) + \psi \int_0^t e_i(\tau)\tau^{\alpha-1} d\tau,$$

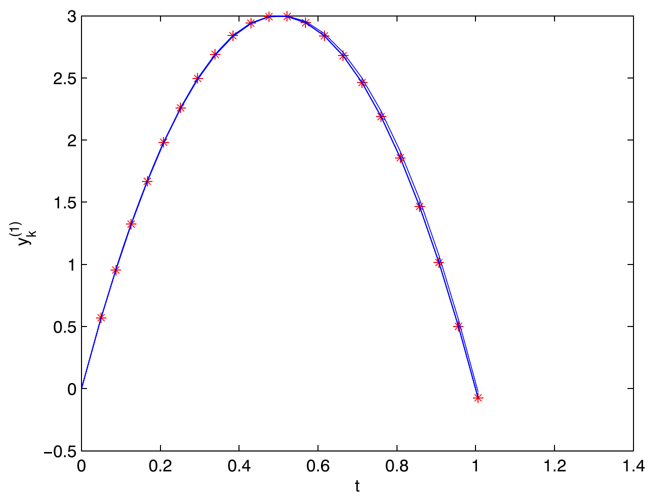

FIGURE 5 The tracking performance of Example 4.3 [Colour figure can be viewed at wileyonlinelibrary.com]

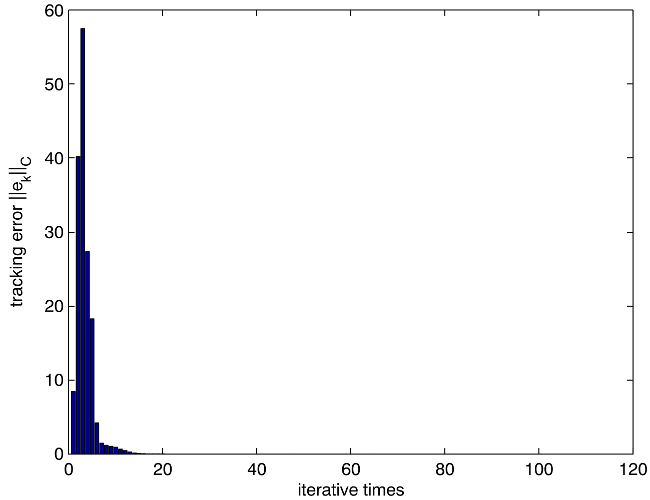

FIGURE 6 The the numerical simulation results of Example 4.3 [Colour figure can be viewed at wileyonlinelibrary.com]

where we choose

$$\Gamma = \begin{pmatrix} 5 & 2 \\ 0 & 3 \end{pmatrix}, \quad L = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \quad \psi = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

and given desired reference trajectory as

$$y_d(t) = \begin{pmatrix} 12t(1-t) \\ 10\sin t \end{pmatrix}, t \in [0, 1].$$

It is obviously that the state $x_i \in \mathbb{R}^2$, output $y_i \in \mathbb{R}^2$, and input $u_i \in \mathbb{R}^2$. One can see that $\rho[I - P(t)Q(t)\Gamma(t)] = 0.7 < 1$. Obviously, the output $y_i(t)$ can track the reference trajectory $y_d(t)$ effectively. In specific applications, when the accuracy of the error reaches the requirements, we can stop iterating. Thus, Theorem 3.3 can be applied to Example 4.3.

Figure 5 shows the tracking performance of Example 4.3. The blue line represents the output track of 1st, 15th, 30th, … , and 90th runs, and the red star represents reference trajectory.

Figure 6 shows the numerical simulation results, and the 100th iteration error is 2.79×10^{-14}.

5 | CONCLUSIONS

This article studies iterative learning control for nonlinear and linear conformable fractional differential equations. The learning updating law is proposed to derive the convergence results for such type equations varying with the initial state is (not) coincident with the desired initial state. The validity of the theoretical results is verified by numerical examples.

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REFERENCES

1. Khalil R, Al Horani M, Yousef A, Sababheh M. A new definition of fractional derivative. *J Comput Appl Math*. 2014;264:65-70.
2. Kilbas A, Srivastava H, Trujillo J. *Theory and Applications of Fractional Differential Equations*, Math. Studies.New York: North-Holland; 2006.
3. Chung W. Fractional Newton mechanics with conformable fractional derivative. *J Comput Appl Math*. 2015;290:150-158.
4. Kumar D, Seadawy AR, Joardar AK. Modified Kudryashov method via new exact solutions for some conformable fractional differential equations arising in mathematical biology. *Chinese J Phys*. 2018;56:75-85.
5. Zhou Y. Attractivity for fractional evolution equations with almost sectorial operators. *Fract Calc Appl Anal*. 2018;21:786-800.
6. Li M, Wang J. Exploring delayed Mittag-Leffler type matrix functions to study finite time stability of fractional delay differential equations. *Appl Math Comput*. 2018;324:254-265.
7. Ahmad B, Luca R. Existence of solutions for sequential fractional integro-differential equations and inclusions with nonlocal boundary conditions. *Appl Math Comput*. 2018;339:516-534.
8. Abdeljawad T. On conformable fractional calculus. *J Comput Appl Math*. 2015;279:57-66.
9. Li M, Wang J, O'Regan D. Existence and Ulam's stability for conformable fractional differential equations with constant coefficients. *Bull Malays Math Sci Soc*. 2017:1-22. https://doi.org/10.1007/s40840-017-0576-7
10. Ünal E, Gökdoğan A. Solution of conformable fractional ordinary differential equations via differential transform method. *Optik*. 2017;128:264-273.
11. Bayour B, Torres DFM. Existence of solution to a local fractional nonlinear differential equation. *J Comput Appl Math*. 2017;312:127-133.
12. Pospíšil M, Pospíšilová Škripková L. Sturms theorems for conformable fractional differential equations. *Math Commun*. 2016;21:273-281.
13. Uchiyama M. Formulation of high-speed motion pattern of a mechanical arm by trial. *Trans Soc Inst Cont Eng*. 1978;14:706-712.
14. Arimoto S, Kawamura S. Bettering operation of robots by learning. *J Robotic Syst*. 1984;1:123-140.
15. Liu S, Debbouche A, Wang J. On the iterative learning control for stochastic impulsive differential equations with randomly varying trial lengths. *J Comput Appl Math*. 2017;312:47-57.
16. Liu S, Debbouche A, Wang J. ILC method for solving approximate controllability of fractional differential equations with noninstantaneous impulses. *J Comput Appl Math*. 2018;339:343-355.
17. Liang C, Wang J, Fečkan M. A study on ILC for linear discrete systems with single delay. *J Differ Equ Appl*. 2018;24:358-374.
18. Lan YH. Iterative learning control with initial state learning for fractional order nonlinear systems. *Comput Math Appl*. 2012;64:3210-3216.
19. Yu X, Debbouche A, Wang J. On the iterative learning control of fractional impulsive fractional evolution equations in Banach spaces. *Math Meth Appl Sci*. 2017;40:6061-6069.
20. Huang D, Li X, Xu J, Xu C, He W. Iterative learning control of inhomogeneous distributed parameter systems-frequency domain design and analysis. *Syst Cont Lett*. 2014;72:22-29.
21. Luo D, Wang J, Shen D. Learning formation control for fractional-order multi-agent systems. *Math Meth Appl Sci*. 2018;41:5003-5014.
22. Wang X, Wang J, Liu S. Iterative learning control for linear conformable fractional differential equations. In: 2018 IEEE 7th Data Driven Control and Learning Systems Conference. May 25-27; Enshi, Hubei. in press.
23. Ortega JM, Rheinboldt WC. *Iterative Solution of Nonlinear Equations in Several Variables*. New York: Academic Press; 1970.
24. Bainov D, Simeonov P. *Integral Inequalities and Applications*. Dordrecht: Kluwer Academic Publishers; 1992.

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