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RESEARCH ARTICLE

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Sampled-data iterative learning control for continuous-time nonlinear systems with iteration-varying lengths

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Summary

In this work, sampled-data iterative learning control (ILC) method is extended to a class of continuous-time nonlinear systems with iteration-varying trial lengths. In order to propose a unified ILC algorithm, the tracking errors will be redefined when the trial length is shorter or longer than the desired one. Based on the modified tracking errors, 2 sampled-data ILC schemes are proposed to handle the randomly varying trial lengths. Sufficient conditions are derived rigorously to guarantee the convergence of the nonlinear system at each sampling instant. To verify the effectiveness of the proposed ILC laws, simulations for a nonlinear system are performed. The simulation results show that if the sampling period is set to be small enough, the convergence of the learning algorithms can be achieved as the iteration number increases.

KEYWORDS

initial state condition, iteration learning control, iteration-varying lengths, iteratively moving average operator, relative degree, sampled-data

1 | INTRODUCTION

Iterative learning control (ILC) is an effective control strategy for systems operating on a finite interval repeatedly. It can be applied to improve the control performance by learning from the previous control experience. That is, the control and tracking information of previous iterations can be fully employed to generate the control signal for the current iteration. In this case, the tracking performance can be gradually improved as the iteration number increases. This ILC was first proposed in 1984 by Arimoto et al[1] for precise robot tracking and now has been fruitful in both the underlying theory and experimental applications in other works.[2-4] Over the past 3 decades, ILC has been extensively investigated in various issues including robust design,[5] distributed algorithms,[6,7] monotonic convergence,[8] and networked configuration.[9,10] Typical applications of ILC can be found in various robots[11,12] and industrial devices.[13,14]

In classic ILC, it requires that every execution (trial, iteration, pass) must be completed in a fixed time duration. However, in many practical control systems, this requirement may not hold due to the limitations of control objects, system constraints, or safety problems. For instance, when stroke patients do the rehabilitation via functional electrical stimulation, sometimes the process has to be terminated midway if the patients do not feel well.[15-17] However, this incomplete treatment information is also very helpful to the upcoming treatments. From ILC point of view, this example forms an ILC problem with iteration-varying trial lengths. Another example is the gait problems of humanoid robots discussed in the works of Longman and Mombaur,[18] in which the walking motion is divided into phases defined by foot strike times and the durations of phases are different from cycle to cycle during the learning process. This also will lead to a nonuniform trial length problem. Furthermore, system input/output constraints will also imply nonrepeatable trial lengths in ILC design, such as the lab-scale gantry crane system given in the work of Guth et al.[19] When the output constraints are violated, the load is wound up and the trial has to be terminated, which makes the trial lengths different from trial to trial. Generally, nonuniform trial length problem may often happen when applying ILC to practical applications in consideration of operation safety. Hence, it is important to investigate ILC for systems with iteration-varying trial lengths.

In the past few years, several works related to ILC with variable trial lengths have been published.[16-28] For instance, Li et al[21-23] introduce a newly defined stochastic variable and an iteration-average operator into ILC algorithm to deal with the randomly varying trial lengths of both discrete-time linear and continuous-time nonlinear systems, where the convergence of the tracking error is derived in the sense of mathematical expectation. Motivated by the works of Li et al,[21-23] Shen et al[24] extend ILC with variable pass lengths to a class of discrete-time nonlinear systems. Furthermore, Shen et al[25,26] derive the convergence of a P-type updating law with nonuniform trial lengths in the sense of almost sure and mean square. Additionally, Seel et al[20] mainly focus on the monotonic convergence property of ILC with variable pass lengths, where the proposed ILC law has been implemented in functional electrical stimulation–based treatment for stroke patients in their other works.[16,17] Shi et al[27] further extend the concept of ILC with iteration-varying trial lengths to a class of stochastic systems. Moreover, Liu and Liu[28] discuss ILC with randomly varying trial lengths under the framework of Lyapunov theory. However, it is worthy to mention that most existing literature concentrate on discrete-time or continuous-time systems and major results are derived for linear systems. Taking the practical systems and computer-aided design methodology into account, it is natural and meaningful to consider the sampled-data control and investigate how sampled-data ILC with nonuniform trial lengths works. However, to our best knowledge, no paper has been reported on this topic. In literature, there are many works focusing on ILC design for sampled-data systems[29-33] since most of the control plants in practice are continuous-time systems, but the digital implementation is discrete-time. Nevertheless, these works only consider the sampled-data ILC with identical trial length. Note that the uniform trial length case can be regarded as a special case of the uniform trial lengths problem, thus the results derived in this paper would include the existing results on sampled-data ILC as special cases. In other words, our results extend the application range of sampled-data ILC.

In this paper, we aim at sampled-data ILC design for continuous-time nonlinear systems with randomly varying trial lengths. To deal with the iteration-varying trial lengths, 2 sampled-data ILC schemes are proposed based on the modified tracking errors that have been redefined when the trial length is shorter or longer than the desired one. Sufficient conditions are derived rigorously to guarantee the convergence of the nonlinear system at each sampling instant. To verify the effectiveness of the proposed ILC laws, simulations for a nonlinear system are also performed. The simulation results show that if the sampling period is set to be small enough, the convergence of the learning algorithms can be achieved as the iteration number increases.

The main contributions of this work are summarized as follows.

- We provide the first result on sampled-data ILC for continuous-time nonlinear systems with iteration-varying lengths. This work fulfills the gap for completing the explorations on ILC under iteration-varying lengths.
- We present an in-depth convergence analysis of the generic and iteration-moving-averaged PD-type ILC update laws. The sufficient conditions for asymptotical convergence are derived rigorously by using contraction mapping method.
- We consider the general relative degree for nonlinear systems and its effect on the convergence. The impact of initial state deviations on the final tracking performance is also discussed.

This paper is organized as follows. Section 2 presents the descriptions of sampled-data ILC methodology to a class of nonlinear systems with randomly varying trial lengths and higher relative degree. In Section 3, two ILC algorithms are proposed associated with their convergence analysis for the identical initial condition case. In Section 4, the influence of varying initial states is discussed for the proposed ILC algorithms. Furthermore, an illustrative example is given in Section 5.

Notation. $\|\cdot\|$ denotes the Euclidean norm. $E(\cdot)$ denotes the mathematical expectation. $\|\theta(i)\|_\lambda = \sup_{i\in\Omega}\alpha^{-\lambda i}E\|\theta(i)\|$ indicates the λ norm of a vector $\theta(i)$, where $\lambda > 0, \alpha > 1$, and Ω is a finite set of i.

2 | PROBLEM FORMULATION

2.1 | System description

Consider the following continuous-time nonlinear system:

$$\begin{aligned} \dot{x}_k(t) &= f\left(x_k(t)\right) + B\left(x_k(t)\right) u_k(t) \\ y_k(t) &= g\left(x_k(t)\right), \end{aligned} \tag{1}$$

where $k = 0, 1, \ldots$ denotes the iteration index, $t \in [0, T_k]$ denotes the time index, and T_k is the actual trial length of the kth iteration. Moreover, $x_k(t) \in \mathbb{R}^n$, $u_k(t) \in \mathbb{R}^p$, and $y_k(t) \in \mathbb{R}^q$ are the state, the control input, and the output of system (1), respectively. The nonlinear functions $f(\cdot) \in \mathbb{R}^n$, $B(\cdot) \in \mathbb{R}^{n\times p}$, and $g(\cdot) = [g_1(\cdot), g_2(\cdot), \ldots, g_q(\cdot)]^T \in \mathbb{R}^q$ are smooth in their domain of definition.

Let h be the sampling period of the sampler and $n_k = [T_k/h]$ be the actual sampling number in the kth iteration. The notation $[\iota]$ means the largest integer less or equal to ι. Furthermore, the desired trajectory is denoted by $y_d(t)$ and assumed to be realizable, where $t \in [0, T_d]$, T_d is the desired length of each iteration and $n_d = [T_d/h]$ is the largest number of desired sampling instants.

The control input is derived from the ILC law that is designed by using the sampling signal. In order to generate a continuous control input, the zero-order holder device is adopted. Then, the continuous-time control signal is taken piecewise constant between the sampling instants

$$u_k(t) = u_k(ih), \tag{2}$$

where $t \in [ih, ih + h)$, $0 \le i \le n_k - 1$.

The control objective in this paper is to design a sampled-data ILC law $u_k(ih)$ such that the output error at each sampling instant satisfies $\lim_{k\to\infty} \|y_d(ih) - y_k(ih)\| = 0$, $0 \le i \le n_k$.

To describe the input-output causal relationship of system (1), we need the derivative notations that are defined as

$$L_f g(x) = \frac{\partial g(x)}{\partial x} f(x)$$

and

$$L_f^j g(x) = L_f\left(L_f^{j-1} g(x)\right) = \frac{\partial\left(L_f^{j-1} g(x)\right)}{\partial x} f(x)$$

$$L_b L_f g(x) = \frac{\partial L_f g(x)}{\partial x} b(x)$$

with $L_f^0 g(x) = g(x)$, where the superscript “0” means no derivative operation.

Definition 1. (See the work of Sun and Wang[30])
The continuous-time nonlinear system (1) has extended relative degree $\{\beta_1, \beta_2, \ldots, \beta_q\}$ for $x(t)$ and the following conditions hold:

1.
$$\int_{ih}^{ih+h} L_{b_r} g_m\left(x(t_1)\right) \mathrm{d}t_1 = 0, 1 \le r \le p, 1 \le m \le q;$$

2.
$$\int_{ih}^{ih+h} \int_{ih}^{t_1} \cdots \int_{ih}^{t_j} L_{b_r} L_f^j g_m(x(t_{j+1}))\mathrm{d}t_{j+1}\cdots\mathrm{d}t_1 = 0,$$
where $1 \le j \le \beta_m - 2$;

3. the $q \times p$ matrix is of full-column rank.

$$\begin{bmatrix} \int_{ih}^{ih+h} \int_{ih}^{t_1} \cdots \int_{ih}^{t_{\beta_1-1}} \left[L_{b_1} L_f^{\beta_1-1} g_1\left(x(t_{\beta_1})\right), \ldots, L_{b_p} L_f^{\beta_1-1} g_1\left(x(t_{\beta_1})\right) \mathrm{d}t_{\beta_1}\cdots\mathrm{d}t_1 \right. \\ \vdots \\ \int_{ih}^{ih+h} \int_{ih}^{t_1} \cdots \int_{ih}^{t_{\beta_q-1}} \left[L_{b_1} L_f^{\beta_q-1} g_q\left(x(t_{\beta_q})\right), \ldots, L_{b_p} L_f^{\beta_q-1} g_q\left(x(t_{\beta_q})\right) \mathrm{d}t_{\beta_q}\cdots\mathrm{d}t_1 \right. \end{bmatrix}.$$

From system (1) and Definition 1, we can derive that the mth component of system output at the sampling instant $ih+h$ of the kth iteration is evaluated as

$$\begin{aligned} y_{m,k}(ih+h) = {} & y_{m,k}(ih) + hL_f g_m\left(x_k(ih)\right) + \cdots + \frac{h^{\beta_m-1}}{(\beta_m-1)!} L_f^{\beta_m-1} g_m\left(x_k(ih)\right) \\ & + \int_{ih}^{ih+h}\int_{ih}^{t_1}\cdots\int_{ih}^{t_{\beta_m-1}} L_f^{\beta_m} g_m\left(x_k\left(t_{\beta_m}\right)\right) \mathrm{d}t_{\beta_m}\cdots \mathrm{d}t_1 \\ & + \int_{ih}^{ih+h}\int_{ih}^{t_1}\cdots\int_{ih}^{t_{\beta_m-1}} \left[L_{b_1}L_f^{\beta_m-1} g_m\left(x_k\left(t_{\beta_m}\right)\right), \ldots, L_{b_p}L_f^{\beta_m-1} g_m\left(x_k\left(t_{\beta_m}\right)\right)\right] \mathrm{d}t_{\beta_m}\cdots \mathrm{d}t_1 u_k(ih). \end{aligned} \tag{3}$$

It indicates that the output $y_{m,k}(ih+h)$ is obtained from the control input $u_k(ih)$. Thus, $\{u_k(ih), y_{m,k}(ih+h), 1 \le m \le q\}$ is a pair of dynamically related cause and effect.

Assumptions are as follows.

Assumption 1. For any realizable reference trajectory $y_d(t)$, there exist a suitable initial state $x_d(0)$ and unique input $u_d(t) \in \mathbb{R}^p$ such that

$$\begin{aligned} \dot{x}_d(t) &= f\left(x_d(t)\right) + B\left(x_d(t)\right) u_d(t) \\ y_d(t) &= g\left(x_d(t)\right), \end{aligned} \tag{4}$$

where $t \in [0, T_d]$ and $u_d(t)$ is uniformly bounded for all $t \in [0, T_d]$.

Assumption 2. For each fixed $x_k(0)$, the mappings S (a mapping from $(x_k(0), u_k(t), t \in [0, T_k])$ to $(x_k(t), t \in [0, T_k])$) and O (a mapping from $(x_k(0), u_k(t), t \in [0, T_k])$ to $(y_k(t), t \in [0, T_k])$) are one to one.

Assumption 3. The system has extended relative degree $\{\beta_1, \beta_2, \ldots, \beta_q\}$ for $x(t), t \in [0, T_d]$.

Assumption 4. The functions $f(\cdot)$, $g(\cdot)$, $B(\cdot)$, $L_f^j g_m(\cdot)$, and $L_{b_r} L_f^{\beta_m-1} g_m(\cdot)$, $0 \le j \le \beta_m$, $1 \le m \le q$, $1 \le r \le p$ are globally Lipishitz in x on $[0, T_d]$. The Lipishitz constants are denoted by l_f, l_g, l_B, l_{Lf}, and L_{bf}, respectively.

Remark 1. Assumption 1 presents the realizability of the reference trajectory, which is widely used in the existing literature for nonlinear systems. Indeed, this assumption can be guaranteed by Assumption 2, while Assumption 2 further implies the existence and uniqueness of the solution to system (1). Assumption 3 describes the extend relative degree, where the latter is defined above. Assumption 4 is imposed to limit the nonlinearities so that the Gronwall inequality and contraction mapping method can be applied to derive the strict convergence analysis.

Assumption 5. The initial state conditions are identical in each iteration, ie, $x_k(0) = x_d(0), \forall k$.

Assumption 6. The initial states for any iteration are bounded, ie, $\|x_d(0) - x_k(0)\| \le \epsilon$, where ϵ is a positive constant.

Remark 2. Assumption 5 is imposed to ensure perfect tracking performance. However, in many practical applications, the identical initial states in each iteration cannot hold, because the value of initial state $x_k(0)$ may not be reset accurately in each iteration. Thus, Assumption 6 is given to relax such assumption to the case that the initial state deviations are bounded. Clearly, Assumption 6 holds generally for most practical systems. Assumptions 5 and 6 will be addressed in Sections 3 and 4, respectively.

2.2 | Problem description

In this paper, we consider the continuous-time nonlinear system with iteration-varying lengths. In order to improve the performance by ILC, it is necessary to suitably address the randomness of the actual trial length T_k in each iteration. Without loss of generality, assume that there exist a minimal trial length $T_{\min}$ and a maximal trial length $T_{\max}$, and the actual trial length in each iteration varies among $[T_{\min}, T_{\max}]$. Besides, the desired trial length T_d satisfies the condition $T_{\min} \le T_d \le T_{\max}$.

Let T_k be a stochastic variable and the probability of the output occurrence at time t be $p(t)$, then the probability distribution function of T_k is

$$\mathbb{P}(T_k \leq t) = \begin{cases} 0, & t \in [0, T_{\min}) \\ p(t), & t \in [T_{\min}, T_{\max}] \\ 1, & t \in (T_{\max}, +\infty], \end{cases}$$

where $0 \leq p(t) \leq 1$. Besides, the output at any time $t < t'$ is available in an iteration if the output at time t' is available in the same iteration. Similar to the work of Liu and Liu,[28] we can define the general form of this probability distribution without prior information

$$p(t) = \begin{cases} 1, & t \in [0, T_{\min}) \\ p_{\max} + \int_t^{T_{\max}} \varsigma(\tau)\mathrm{d}\tau, & t \in [T_{\min}, T_{\max}] \end{cases}$$

$$\int_{T_{\min}}^{T_{\max}} \varsigma(\tau)\mathrm{d}\tau = 1 - p_{\max},$$

where $\varsigma(\tau)$ is a probability density function, $p_{\max} > 0$ is the probability of the event that the trial length is $T_{\max}$, and thus the probability $p(t)$ satisfies the condition $0 < p_{\max} \leq p(t) \leq 1$. Furthermore, compared with the Gaussian probability distribution in the work of Shi et al,[27] $p(t)$ is of no prior form in this work. As a result, the formulation of randomly varying lengths in the aforementioned work[27] can be seen as a special case of the formulation in this paper.

Obviously, there are 2 cases for sampling numbers to be addressed, ie, $n_k < n_d$ and $n_k \geq n_d$. For the former case, the kth iteration would end before the desired trial length is achieved, and the outputs on the interval $(n_k, n_d]$ are missing, which are not available for updating. For the latter case, the kth iteration will still run up to the time instant n_k instead of stopping at n_d. It is observed that the data after the time instant n_d are redundant and useless for learning. Without loss of generality, we could simply let the latter case be $n_k = n_d$.

Then, let $\mathbf{1}(ih \leq n_k h), i \in [0, n_d]$ be a stochastic variable taking binary values 0 and 1. Here, $\mathbf{1}(ih \leq n_k h) = 1$ denotes the event that the control process can last beyond the sampling time instant ih, which occurs with a probability of $p(ih)$, where $0 < p(ih) < 1$, whereas $\mathbf{1}(ih \leq n_k h) = 0$ denotes the event that the control process cannot continue to the sampling time instant ih, which occurs with a probability of $1 - p(ih)$. It is apparent that $\mathbf{1}(ih \leq n_k h)$ satisfies the Bernoulli distribution, thus we can derive the expectation $\mathbb{E}\{\mathbf{1}(ih \leq n_k h)\} = 1 \cdot p(ih) + 0 \cdot (1 - p(ih)) = p(ih)$.

Therefore, we can define a modified tracking error as follows:

$$e_k^*(ih) = \begin{cases} e_k(ih) & 0 \leq i \leq n_k \\ 0 & (n_k + 1) \leq i \leq n_d, \end{cases} \tag{5}$$

where $e_k(ih) \triangleq y_d(ih) - y_k(ih)$ is the original tracking error.

Then, (5) could be reformulated as

$$e_k^*(ih) = \mathbf{1}(ih \leq n_k h) e_k(ih). \tag{6}$$

Lemma 1. (See the work of Shen et al[24])
Let ξ be a Bernoulli binary random variable with $\mathbb{P}(\xi = 1) = \bar{\xi}$ and $\mathbb{P}(\xi = 0) = 1 - \bar{\xi}$. If there exists a positive matrix Ψ, the equality $\mathbb{E}\|I - \xi\Psi\| = \|I - \bar{\xi}\Psi\|$ holds if and only if one of the following conditions is satisfied: (i) $\bar{\xi} = 1$ or $\bar{\xi} = 0$; (ii) $0 < \bar{\xi} < 1$; and (iii) $0 < \Psi \leq I$.

3 | SAMPLED-DATA ILC DESIGN AND CONVERGENCE ANALYSIS

Two ILC laws with the modified tracking error are introduced in this section, and convergence analyses are addressed.

3.1 | Generic proportional-derivative (PD)–type ILC scheme

The generic PD-type ILC is given by

$$u_{k+1}(ih) = u_k(ih) + K_P e_k^*((i+1)h) + K_D \left(e_k^*((i+1)h) - e_k^*(ih) \right), \tag{7}$$

where $0 \leq i \leq n_d - 1$, and $K_P \in \mathbb{R}^{p \times q}$ and $K_D \in \mathbb{R}^{p \times q}$ are proportional and derivative learning gains, respectively. These gains will be defined in the following.

Theorem 1. *Consider the continuous-time nonlinear system (1) with Assumptions 1 to 5. Let the PD-type ILC law (7) be applied with learning gains K_P and K_D satisfying*

$$\sup_k \sup_i \mathbb{E}\,\|I - N_k(ih)\psi_k(ih)\| \leq \sigma < 1, \tag{8}$$

where $N_k(ih) = (K_P + K_D)\mathbf{1}((i+1)h \leq n_k h)$ and

$$\psi_k(ih) = \begin{bmatrix} \int_{ih}^{ih+h}\int_{ih}^{t_1}\cdots\int_{ih}^{t_{\beta_1-1}}\left[L_{b_1}L_f^{\beta_1-1}g_1\left(x\left(t_{\beta_1}\right)\right),\ \ldots,L_{b_p}L_f^{\beta_1-1}g_1\left(x\left(t_{\beta_1}\right)\right)\mathrm{d}t_{\beta_1}\cdots\mathrm{d}t_1\right. \\ \vdots \\ \int_{ih}^{ih+h}\int_{ih}^{t_1}\cdots\int_{ih}^{t_{\beta_q-1}}\left[L_{b_1}L_f^{\beta_q-1}g_q\left(x\left(t_{\beta_q}\right)\right),\ \ldots,L_{b_p}L_f^{\beta_q-1}g_q\left(x\left(t_{\beta_q}\right)\right)\mathrm{d}t_{\beta_q}\cdots\mathrm{d}t_1\right. \end{bmatrix}.$$

If the sampling period h is chosen small enough, then the system output $y_k(ih)$ converges to $y_d(ih)$ for all $i \in [1, n_d]$ as $k \to \infty$.

Proof. Define $\delta u_k(\cdot) = u_d(\cdot) - u_k(\cdot)$ and $\delta x_k(\cdot) = x_d(\cdot) - x_k(\cdot)$. It follows from (6) and (7) that

$$\begin{aligned}\delta u_{k+1}(ih) &= \delta u_k(ih) - K_P e_k^*((i+1)h) - K_D\left(e_k^*((i+1)h) - e_k^*(ih)\right)\\ &= \delta u_k(ih) - (K_P+K_D)e_k^*((i+1)h) + K_D e_k^*(ih)\\ &= \delta u_k(ih) - N_k(ih)e_k((i+1)h) + M_k(ih)e_k(ih),\end{aligned} \tag{9}$$

where $N_k(ih) = (K_P + K_D)\mathbf{1}((i+1)h \leq n_k h)$ and $M_k(ih) = K_D\mathbf{1}(ih \leq n_k h)$.

From (3), the mth component of tracking error at the sampling time instant $(i+1)h$ can be expressed as

$$\begin{aligned}e_{m,k}((i+1)h) &= y_{m,d}((i+1)h) - y_{m,k}((i+1)h)\\ &= e_{m,k}(ih) + \upsilon_{m,k}(ih) + \omega_{m,k}(ih) + \psi_{m,k}(ih)\delta u_k(ih),\end{aligned} \tag{10}$$

where

$$\begin{aligned}e_{m,k}(ih) &= g_m(x_d(ih)) - g_m(x_k(ih)),\\ \upsilon_{m,k}(ih) &= h\left[L_f g_m(x_d(ih)) - L_f g_m(x_k(ih))\right] + \cdots + \frac{h^{\beta_m-1}}{(\beta_m-1)!}\left[L_f^{\beta_m-1}g_m(x_d(ih)) - L_f^{\beta_m-1}g_m(x_k(ih))\right]\\ \omega_{m,k}(ih) &= \int_{ih}^{ih+h}\int_{ih}^{t_1}\cdots\int_{ih}^{t_{\beta_m-1}}\left[L_f^{\beta_m}g_m\left(x_d(t_{\beta_m})\right) - L_f^{\beta_m}g_m\left(x_k(t_{\beta_m})\right)\right]\mathrm{d}t_{\beta_m}\cdots\mathrm{d}t_1\\ &\quad+\int_{ih}^{ih+h}\int_{ih}^{t_1}\cdots\int_{ih}^{t_{\beta_m-1}}\left[L_{b_1}L_f^{\beta_m-1}g_m\left(x_d(t_{\beta_m})\right),\ \ldots,L_{b_p}L_f^{\beta_m-1}g_m\left(x_d(t_{\beta_m})\right)\right]\\ &\quad-\left[L_{b_1}L_f^{\beta_m-1}g_m\left(x_k(t_{\beta_m})\right),\ \ldots,L_{b_p}L_f^{\beta_m-1}g_m\left(x_k(t_{\beta_m})\right)\right]\mathrm{d}t_{\beta_m}\cdots\mathrm{d}t_1 u_d(ih)\\ \psi_{m,k}(ih) &= \int_{ih}^{ih+h}\int_{ih}^{t_1}\cdots\int_{ih}^{t_{\beta_m-1}}\left[L_{b_1}L_f^{\beta_m-1}g_m\left(x_k(t_{\beta_m})\right),\ \ldots,L_{b_p}L_f^{\beta_m-1}g_m\left(x_k(t_{\beta_m})\right)\right]\mathrm{d}t_{\beta_m}\cdots\mathrm{d}t_1.\end{aligned}$$

The tracking error at time instant $(i+1)h$ can be written as

$$e_k((i+1)h) = e_k(ih) + \upsilon_k(ih) + \omega_k(ih) + \psi_k(ih)\delta u_k(ih), \tag{11}$$

where

$$\begin{aligned}e_k(ih) &= \left[e_{1,k}(ih),\ \ldots,e_{q,k}(ih)\right]^T,\\ \upsilon_k(ih) &= \left[\upsilon_{1,k}(ih),\ \ldots,\upsilon_{q,k}(ih)\right]^T,\\ \omega_k(ih) &= \left[\omega_{1,k}(ih),\ \ldots,\omega_{q,k}(ih)\right]^T,\\ \psi_k(ih) &= \left[\psi_{1,k}^T(ih),\ \ldots,\psi_{q,k}^T(ih)\right]^T.\end{aligned}$$

Substituting (11) into (9) yields

$$\delta u_{k+1}(ih) = (I - N_k(ih)\psi_k(ih))\,\delta u_k(ih) + (M_k(ih) - N_k(ih))\,e_k(ih) - N_k(ih)\,(\upsilon_k(ih) + \omega_k(ih)). \tag{12}$$

Taking norms to both sides of (12) and applying the Lipschitz condition in Assumption 4, we have

$$\|\delta u_{k+1}(ih)\| \leq \|I - N_k(ih)\psi_k(ih)\|\,\|\delta u_k(ih)\| + l_g\gamma_m\,\|\delta x_k(ih)\| + \gamma_n\,(\|\upsilon_k(ih)\| + \|\omega_k(ih)\|) \tag{13}$$

and

$$
\|v_k(ih)\| \le \gamma_1 \|\delta x_k(ih)\|,
$$
$$
\|\omega_k(ih)\| \le \gamma_2 \left\| \begin{bmatrix} \int_{ih}^{ih+h} \int_{ih}^{t_1} \cdots \int_{ih}^{t_{\beta_1-1}} \left\|\delta x_k(t_{\beta_1})\right\| dt_{\beta_1} \cdots dt_1 \\ \vdots \\ \int_{ih}^{ih+h} \int_{ih}^{t_1} \cdots \int_{ih}^{t_{\beta_q-1}} \left\|\delta x_k(t_{\beta_q})\right\| dt_{\beta_q} \cdots dt_1 \end{bmatrix} \right\|,
$$

where $\gamma_1 = \max_{1\le m\le q}\left\{\frac{h}{1!}+\cdots+\frac{h^{\beta_m-1}}{(\beta_m-1)!}\right\} l_{Lf}$, $\gamma_2 = l_{Lf} + pl_{bf}\rho_{ud}$, $\rho_{ud} = \sup_{0\le i\le n_k-1}\|u_d(ih)\|$, γ_m is the norm bound for $(M_k(ih) - N_k(ih))$ and γ_n is the norm bound for $N_k(ih)$.

From (1) and (4), we can obtain

$$
\delta x_k(t) = \int_{ih}^{t} [f(x_d(\tau)) - f(x_k(\tau))]\,\mathrm{d}\tau + \int_{ih}^{t} [B(x_d(\tau))\,u_d(\tau) - B(x_k(\tau))\,u_k(\tau)]\,\mathrm{d}\tau, \tag{14}
$$

where $t \in [ih, ih+h]$. Then, taking the norms and applying Bellman-Gronwall's lemma to (14) results in that

$$
\begin{aligned}
&\|\delta x_k(t)\| \le \|\delta x_k(ih)\|\, e^{\gamma_3(t-ih)} + \int_{ih}^{t} e^{\gamma_3(t-s)}\gamma_B \|\delta u_k(s)\|\,\mathrm{d}s, \\
&\|\delta x_k(t)\| \le \gamma_4 \|\delta x_k(ih)\| + \gamma_5 \|\delta u_k(ih)\|,
\end{aligned} \tag{15}
$$

where $\gamma_3 = l_f + l_B\rho_{ud}$, $\gamma_4 = e^{\gamma_3 h}$, $\gamma_5 = \frac{\gamma_B(\gamma_4-1)}{\gamma_3}$, and γ_B is the norm bound for $B(x_k(t))$. Moreover, we can obtain

$$
\|\delta x_k(ih)\| \le \gamma_4 \|\delta x_k((i-1)h)\| + \gamma_5 \|\delta u_k((i-1)h)\|. \tag{16}
$$

According to Assumption 5, ie, $x_k(0) = x_d(0)$ for all k, it has

$$
\|\delta x_k(ih)\| \le \gamma_5 \sum_{j=0}^{i-1} \gamma_4^{i-1-j} \|\delta u_k(jh)\|. \tag{17}
$$

Then, (13) can be rewritten as

$$
\|\delta u_{k+1}(ih)\| \le \bar{\rho}_k(ih)\, \|\delta u_k(ih)\| + \gamma_6 \|\delta x_k(ih)\|, \tag{18}
$$

$$
\|\delta u_{k+1}(ih)\| \le \bar{\rho}_k(ih)\, \|\delta u_k(ih)\| + \gamma_5\gamma_6 \sum_{j=0}^{i-1} \gamma_4^{i-1-j} \|\delta u_k(jh)\|, \tag{19}
$$

where $\bar{\rho}_k(ih) = \rho_k(ih) + \gamma_2\gamma_5\gamma_h\gamma_n$, $\rho_k(ih) = \|I - N_k(ih)\psi_k(ih)\|$, $\gamma_h = \max\left\{\frac{h^{\beta_1}}{\beta_1!}, \ldots, \frac{h^{\beta_q}}{\beta_q!}\right\}$, and $\gamma_6 = (\gamma_1 + \gamma_2\gamma_4\gamma_h)\gamma_n + l_g\gamma_m$. Clearly, sufficiently small sampling period h yields arbitrarily small γ_h.

Furthermore, taking mathematical expectation to both sides of (19), we have

$$
\begin{aligned}
\mathbb{E}\,\|\delta u_{k+1}(ih)\| &\le \mathbb{E}\left(\bar{\rho}_k(ih)\, \|\delta u_k(ih)\|\right) + \mathbb{E}\left(\gamma_5\gamma_6 \sum_{j=0}^{i-1} \gamma_4^{i-1-j} \|\delta u_k(jh)\|\right) \\
&\le \mathbb{E}\left(\bar{\rho}_k(ih)\right) \mathbb{E}\,\|\delta u_k(ih)\| + \gamma_5\gamma_6 \sum_{j=0}^{i-1} \gamma_4^{i-1-j} \mathbb{E}\,\|\delta u_k(jh)\|,
\end{aligned} \tag{20}
$$

where $\mathbb{E}(\bar{\rho}_k(ih)) = \mathbb{E}(\rho_k(ih)) + \gamma_2\gamma_5\gamma_h\gamma_n$ and $\mathbb{E}(\rho_k) = \mathbb{E}\|I - N_k(ih)\psi_k(ih)\|$.

Multiplying both sides of (20) with $\alpha^{-\lambda i}$ and taking supremum for all time instants i yields

$$
\sup_i \alpha^{-\lambda i}\mathbb{E}\,\|\delta u_{k+1}(ih)\| \le \sup_i \mathbb{E}(\bar{\rho}_k) \sup_i \alpha^{-\lambda i}\mathbb{E}\,\|\delta u_k(ih)\| + \gamma_5\gamma_6 \sup_i \alpha^{-\lambda i} \sum_{j=0}^{i-1} \gamma_4^{i-1-j} \mathbb{E}\,\|\delta u_k(jh)\|. \tag{21}
$$

Let $\alpha > \gamma_4$, then we can drive that

$$\begin{aligned}
\sup_i \alpha^{-\lambda i}\sum_{j=0}^{i-1}\gamma_4^{i-1-j}\mathbb{E}\,\|\delta u_k(jh)\| &\leq \sup_i \alpha^{-\lambda i}\sum_{j=0}^{i-1}\alpha^{i-1-j}\mathbb{E}\,\|\delta u_k(jh)\| \\
&\leq \alpha^{-1}\sup_i \sum_{j=0}^{i-1}\left(\sup_j \left(\alpha^{-\lambda j}\mathbb{E}\,\|\delta u_k(jh)\|\right)\alpha^{-(\lambda-1)(i-j)}\right) \\
&\leq \alpha^{-1}\|\delta u_k(ih)\|_\lambda \sup_i \sum_{j=0}^{i-1}\alpha^{-(\lambda-1)(i-j)} \\
&\leq \eta_d \cdot \|\delta u_k(ih)\|_\lambda,
\end{aligned} \tag{22}$$

where $\eta_d = \frac{1-\alpha^{-(\lambda-1)n_d}}{\alpha^\lambda-\alpha}$.

Substituting (22) into (21) implies that

$$\|\delta u_{k+1}(ih)\|_\lambda \leq \sup_i \mathbb{E}\left(\bar{\rho}_k(ih)\right)\|\delta u_k(ih)\|_\lambda + \gamma_5\gamma_6\eta_d\|\delta u_k(ih)\|_\lambda. \tag{23}$$

Let

$$\mu = \sup_k \sup_i \mathbb{E}\left(\bar{\rho}_k(ih)\right)$$

and

$$\kappa = \gamma_5\gamma_6,$$

we have

$$\|\delta u_{k+1}(ih)\|_\lambda \leq (\mu + \kappa\eta_d)\|\delta u_k(ih)\|_\lambda. \tag{24}$$

Let $\alpha > \max\{1, \gamma_4\}$, then it is possible to choose a sufficiently small sampling period h and a sufficiently large λ such that

$$\kappa\eta_d = \kappa\frac{1-\alpha^{-(\lambda-1)n_d}}{\alpha^\lambda - \alpha} \tag{25}$$

is arbitrarily small. Thus, if (8) holds $\forall i$, there exist sufficiently small h and sufficiently large λ such that

$$\mu + \kappa\eta_d \leq \zeta < 1.$$

Then, it is guaranteed that

$$\lim_{k\to\infty}\|\delta u_k(ih)\|_\lambda = 0, \forall i.$$

According to the finiteness of i, it follows

$$\lim_{k\to\infty}\mathbb{E}\|\delta u_k(ih)\| = 0, \forall i.$$

Noticing $\|\delta u_k(ih)\| \geq 0$, we obtain

$$\lim_{k\to\infty}\|\delta u_k(ih)\| = 0, \forall i.$$

Then, it is easy to conclude the results $\lim_{k\to\infty}\delta x_k(ih) = 0$ and $\lim_{k\to\infty}e_k(ih) = 0$, $\forall i$. This completes the proof. □

Theorem 1 presents an explicit sufficient condition guaranteeing the asymptotical convergence of the tracking errors at sampling time instants for the generic PD-type sampled-data ILC for nonlinear systems with iteration-varying lengths. The sufficient condition (8) indicates that gains of both P-part and D-part of the update law would affect the convergence. It is worthwhile to mention that the proposed sampled-data ILC is able to work well without an accurate system model. The learning gains can be determined by some approximations while the sampling period is sufficiently small. Noting that mathematical expectation is involved in the convergence condition, we can remove this operator by strengthening the design of learning gains as in the following corollary.

Corollary 1. *Consider the continuous-time nonlinear system (1) and Assumptions 1 to 5. Let the PD-type ILC law (7) be applied with learning gains K_P and K_D satisfying*

$$\begin{gathered}
0 < (K_P + K_D)\psi_k(ih) < I, \\
\sup_k \sup_i \|I - (K_P + K_D)p\left((i+1)h\right)\psi_k(ih)\| < 1.
\end{gathered} \tag{26}$$

Then, the system output $y_k(ih)$ converges to $y_d(ih)$ for all $i \in [1, n_d]$ as $k \to \infty$ if the sampling period h is chosen small enough.

Proof. Applying the results in Lemma 1 to condition (8) in Theorem 1, we can complete the proof of this corollary. □

Remark 3. In Theorem 1, we depict a qualitative expression of the sampling period h to guarantee the asymptotical convergence that the sampling period should be small enough. One may interest in how is the explicit range of the sampling period. Indeed, it is difficult to determine the period because of the nonlinearities in the system and controller. Generally, it is seen from (24) that the sampling period should ensure that $\gamma_2\gamma_5\gamma_h\gamma_n + \kappa\eta_d < 1 - \sigma$. Since we can always select sufficiently large α to make $\kappa\eta_d$ arbitrarily small, we should still ensure $\gamma_2\gamma_5\gamma_h\gamma_n < 1 - \sigma$. In practical applications, we find that a small sampling period implies a better control performance. From this point of view, it is suggested to select the possible smallest sampling period to guarantee the convergence condition and improve the actual tracking performance in the meantime.

3.2 | The modified ILC scheme

An iteratively moving average operator is used in this section to solve the problem of sampled-data ILC for nonlinear systems with iteration-varying lengths. Compared with the iteration-averaging techniques used in the works of Li et al[22,23] and Shi et al,[27] the iteratively moving average operator in this paper only use the information of several previous trials to compensate the absent tracking information. The inherent reason lies in that the data from early iterations may be useless for current input updating, while the information of adjacent iterations would be helpful in correcting the input signals.

Definition 2. (See the work of Li et al[23])
For a sequence $f_{k-r}(\cdot), f_{k-r+1}(\cdot), \ldots, f_k(\cdot)$ with $r \geq 0$, an iteratively moving average operator is defined as

$$\mathcal{A}\{f_k(\cdot)\} \triangleq \frac{1}{r+1}\sum_{j=0}^{r} f_{k-j}(\cdot), \tag{27}$$

where $r+1$ is the size of the moving window.

As a special case, the iteratively moving average operator of the mth component of the vector sequence is represented as

$$\mathcal{A}\left\{f_k^m(\cdot)\right\} \triangleq \frac{1}{r+1}\sum_{j=0}^{r} f_{k-j}^m(\cdot). \tag{28}$$

Design an iteratively moving average operator-based PD-type ILC law as follows:

$$u_{k+1}(ih) = \mathcal{A}\{u_k(ih)\} + K_P{}'\mathcal{A}\left\{e_k^*((i+1)h)\right\} + K_D{}'\mathcal{A}\left\{e_k^*((i+1)h) - e_k^*(ih)\right\}, \tag{29}$$

where $K_P{}' \in \mathbb{R}^{p\times q}$ and $K_D{}' \in \mathbb{R}^{p\times q}$ are proportional and derivative learning gains, respectively. These gains will be determined in the following. In addition, we assume that $u_{-1}(ih) = u_{-2}(ih) = \cdots = u_{-z}(ih) = 0$ without loss of any generality.

Theorem 2. *Consider the continuous-time nonlinear system (1) with Assumptions 1 to 5. Let the PD-type ILC law (29) be applied with learning gains $K_P{}'$ and $K_D{}'$ satisfying*

$$0 \leq \sum_{w=0}^{r} \vartheta_w \leq \sigma < 1, \tag{30}$$

where

$$\vartheta_w \triangleq \frac{1}{r+1}\sup_k \sup_i \mathbb{E}\left\|I - N_{k-w}{}'(ih)\psi_{k-w}(ih)\right\|$$

and $N_{k-w}{}'(ih) = (K_P{}' + K_D{}')\mathbf{1}((i+1)h \leq n_{k-w}h)$. If the sampling period h is chosen small enough, then the system output $y_k(ih)$ converges to $y_d(ih)$ for all $i \in [1, n_d]$ as $k \to \infty$.

Proof. Substituting (6), (27) into (29) yields to

$$u_{k+1}(ih) = \frac{1}{r+1}\sum_{w=0}^{r} u_{k-w}(ih) + \frac{1}{r+1}\sum_{w=0}^{r} N_{k-w}{}'(ih)e_{k-w}((i+1)h) - \frac{1}{r+1}\sum_{w=0}^{r} M_{k-w}{}'(ih)e_{k-w}(ih), \tag{31}$$

where $M_{k-w}'(ih) = K_D'\mathbf{1}(ih \le n_{k-w}h)$ and $N_{k-w}'(ih) = (K_P' + K_D')\mathbf{1}((i+1)h \le n_{k-w}h)$. Then, it follows that

$$\begin{aligned}\delta u_{k+1}(ih) = &\sum_{w=0}^{r}\frac{1}{r+1}\left(I - N_{k-w}'(ih)\psi_{k-w}(ih)\right)\delta u_{k-w}(ih)\\ &+\sum_{w=0}^{r}\frac{1}{r+1}\left(M_{k-w}'(ih) - N_{k-w}'(ih)\right)e_{k-w}(ih)\\ &-\sum_{w=0}^{r}\frac{1}{r+1}N_{k-w}'(ih)\left(\upsilon_{k-w}(ih) + \omega_{k-w}(ih)\right).\end{aligned} \tag{32}$$

Taking norms to both sides of (32) and applying the Lipschitz condition in Assumption 4 yields that

$$\begin{aligned}\|\delta u_{k+1}(ih)\| \le &\sum_{w=0}^{r}\left\|\frac{1}{r+1}\left(I - N_{k-w}'(ih)\psi_{k-w}(ih)\right)\right\|\|\delta u_{k-w}(ih)\| + \frac{1}{r+1}l_g\gamma_m'\sum_{w=0}^{r}\|\delta x_{k-w}(ih)\|\\ &+\frac{1}{r+1}\gamma_n'\sum_{w=0}^{r}\left(\|\upsilon_{k-w}(ih)\| + \|\omega_{k-w}(ih)\|\right),\end{aligned} \tag{33}$$

where γ_m' is the norm bound for $(M_{k-w}'(ih) - N_{k-w}'(ih))$ and γ_n' is the norm bound for $N_{k-w}'(ih)$.

Then, from (17), we have

$$\|\delta x_{k-w}(ih)\| \le \gamma_5\sum_{j=0}^{i-1}\gamma_4^{i-1-j}\|\delta u_{k-w}(jh)\|, \tag{34}$$

$$\|\upsilon_{k-w}(ih)\| + \|\omega_{k-w}(ih)\| \le (\gamma_1 + \gamma_2\gamma_4\gamma_h)\|\delta x_{k-w}(ih)\| + \gamma_2\gamma_5\gamma_h\|\delta u_{k-w}(ih)\|. \tag{35}$$

Combing (33), (34) and (35), we can obtain

$$\|\delta u_{k+1}(ih)\| \le \frac{1}{r+1}\sum_{w=0}^{r}\bar{\rho}_{k-w}'(ih)\|\delta u_{k-w}(ih)\| + \gamma_5\gamma_7\cdot\frac{1}{r+1}\sum_{w=0}^{r}\sum_{j=0}^{i-1}\gamma_4^{i-1-j}\|\delta u_{k-w}(jh)\|, \tag{36}$$

where $\bar{\rho}_{k-w}'(ih) = \rho_{k-w}'(ih) + \gamma_2\gamma_5\gamma_h\gamma_n'$, $\rho_{k-w}'(ih) = \|I - N_{k-w}'(ih)\psi_{k-w}(ih)\|$, and $\gamma_7 = (\gamma_1 + \gamma_2\gamma_4\gamma_h)\gamma_n' + l_g\gamma_m'$.

Taking mathematical expectation to both sides of (36), we can conclude that

$$\begin{aligned}\mathbb{E}\|\delta u_{k+1}(ih)\| &\le \mathbb{E}\left(\frac{1}{r+1}\sum_{w=0}^{r}\bar{\rho}_{k-w}'(ih)\|\delta u_{k-w}(ih)\|\right) + \mathbb{E}\left(\gamma_5\gamma_7\cdot\frac{1}{r+1}\sum_{w=0}^{r}\sum_{j=0}^{i-1}\gamma_4^{i-1-j}\|\delta u_{k-w}(jh)\|\right)\\ &\le \frac{1}{r+1}\sum_{w=0}^{r}\mathbb{E}\left(\bar{\rho}_{k-w}'(ih)\right)\mathbb{E}\|\delta u_{k-w}(ih)\| + \gamma_5\gamma_7\cdot\frac{1}{r+1}\sum_{w=0}^{r}\sum_{j=0}^{i-1}\gamma_4^{i-1-j}\mathbb{E}\|\delta u_{k-w}(jh)\|,\end{aligned} \tag{37}$$

where $\mathbb{E}(\bar{\rho}_{k-w}'(ih)) = \mathbb{E}(\rho_{k-w}'(ih)) + \gamma_2\gamma_5\gamma_h\gamma_n'$ and $\mathbb{E}(\rho_{k-w}'(ih)) = \mathbb{E}\|I - N_{k-w}'(ih)\psi_{k-w}(ih)\|$.

Multiplying both sides of (37) with $\alpha^{-\lambda i}$ and taking supremum for all time instants i, we have

$$\begin{aligned}\sup_i \alpha^{-\lambda i}\mathbb{E}\|\delta u_{k+1}(ih)\| \le &\frac{1}{r+1}\sum_{w=0}^{r}\sup_i\mathbb{E}\left(\bar{\rho}_{k-w}'(ih)\right)\sup_i \alpha^{-\lambda i}\mathbb{E}\|\delta u_{k-w}(ih)\|\\ &+\gamma_5\gamma_7\cdot\frac{1}{r+1}\sup_i \alpha^{-\lambda i}\sum_{w=0}^{r}\sum_{j=0}^{i-1}\gamma_4^{i-1-j}\mathbb{E}\|\delta u_{k-w}(jh)\|\end{aligned} \tag{38}$$

and

$$\sup_i \alpha^{-\lambda i}\sum_{w=0}^{r}\sum_{j=0}^{i-1}\gamma_4^{i-1-j}\mathbb{E}\|\delta u_{k-w}(jh)\| \le \eta_d\sum_{w=0}^{r}\|\delta u_{k-w}(ih)\|_\lambda. \tag{39}$$

Thus, (38) can be rewritten as

$$\|\delta u_{k+1}(ih)\|_\lambda \le \frac{1}{r+1}\sum_{w=0}^{r}\sup_i\mathbb{E}\left(\bar{\rho}_{k-w}'(ih)\right)\|\delta u_{k-w}(ih)\|_\lambda + \gamma_5\gamma_7\cdot\frac{1}{r+1}\eta_d\sum_{w=0}^{r}\|\delta u_{k-w}(jh)\|_\lambda. \tag{40}$$

Define

$$\vartheta_w \triangleq \frac{1}{r+1}\sup_k\sup_i\mathbb{E}\left(\rho_{k-w}'(ih)\right) \tag{41}$$

and

$$\theta \triangleq \gamma_5 \gamma_7 \eta_d \cdot \frac{1}{r+1}. \tag{42}$$

Then, we have

$$\|\delta u_{k+1}(ih)\|_\lambda \leq \left(\sum_{w=0}^{r} (\vartheta_w + \varrho) + \theta \right) \max \left\{ \|\delta u_k(ih)\|_\lambda, \|\delta u_{k-1}(ih)\|_\lambda, \ldots, \|\delta u_{k-r+1}(ih)\|_\lambda \right\}, \tag{43}$$

where $\varrho = \gamma_2 \gamma_5 \gamma_h \gamma_n{}' \cdot \frac{1}{r+1}$. If we choose a λ large enough and the sampling period h small enough, then θ and ϱ can be made sufficiently small and be independent of k.

From (30), it follows that $\sum_{w=0}^{r} (\vartheta_w + \varrho) + \theta < 1$. This further implies that

$$\lim_{k \to \infty} \|\delta u_k(ih)\|_\lambda = 0, \forall i.$$

Therefore,

$$\lim_{k \to \infty} \|\delta u_k(ih)\| = 0, \forall i.$$

It is apparent that $\lim_{k\to\infty} \delta x_k(ih) = 0$ and $\lim_{k\to\infty} e_k(ih) = 0, \forall i$. This completes the proof. □

Theorem 2 presents a parallel result to Theorem 1 for the iteration-moving-average-operator–based algorithm. Since we have employed the information of the previous r iterations, it can be seen from the condition (30) that the convergence depends on the jointed contraction of the involved iterations. Noting that an average (ie, $1/(r+1)$) is added to ϑ_w, the convergence condition of this theorem is generally not stricter than that of Theorem 1.

Corollary 2. *Consider the continuous-time nonlinear system (1) and Assumptions 1 to 5. Let the PD-type ILC law (29) be applied with learning gains $K_P{}'$ and $K_D{}'$ satisfying that*

$$0 < \left(K_P{}' + K_D{}'\right) \psi_{k-w}(ih) < I,$$
$$0 \leq \sum_{w=0}^{r} \vartheta_w{}' < 1,$$

where

$$\vartheta_w{}' \triangleq \frac{1}{r+1} \sup_k \sup_i \left\| I - \left(K_P{}' + K_D{}'\right) p\left((i+1)h\right) \psi_{k-w}(ih) \right\|.$$

Then, the system output $y_k(ih)$ converges to $y_d(ih)$ for all $i \in [1, n_d]$ as $k \to \infty$ if the sampling period h is chosen small enough.

Remark 4. In many practical applications, there may be stochastic disturbances and measurement noises in the process of control. Such disturbances and noises would lead to a large deviation between the actual output and the desired trajectory in some iteration. In such case, if we only use the information from the last iteration, the computed signals may have remarkable deviations. Meanwhile, when considering nonlinear systems, the nonlinearity may further involve a complex updating process. In this paper, we adopt the tracking information from several iterations and make a combination of such information. This is the iteratively moving operator mechanism proposed above. It is believed that such mechanism would behave its advantage in dealing with the disturbances, noises, and uncertainties.

4 | SAMPLED-DATA ILC DESIGN WITH INITIAL VALUE FLUCTUATION

In practical applications, the value of initial state $x_k(0)$ may not be set precisely in each iteration, which leads to an observation that Assumption 5 does not hold. In this section, we will replace Assumption 5 with a relaxed condition Assumption 6 and propose the corresponding stable convergence results.

4.1 | Generic PD-type ILC scheme

Theorem 3. *Consider the continuous-time nonlinear system (1) with Assumptions 1-4 and 6. Let the PD-type ILC law (7) be applied. If the sampling period h is chosen small enough and the learning gains K_P and K_D satisfy*

$$\sup_k \sup_i \mathbb{E}\, \|I - N_k(ih)\psi_k(ih)\| \leq \sigma < 1,$$

then the tracking error would converge to a small zone, whose upper bound is in proportion to ϵ, for all $i \in [1, n_d]$ as $k \to \infty$, ie, $\lim_{k\to\infty} \sup_i \mathbb{E}\|e_k(ih)\| \leq \gamma_e \epsilon$.

Proof. From (16), it follows

$$\|\delta x_k(ih)\| \leq \gamma_5 \sum_{j=0}^{i-1} \gamma_4^{i-1-j} \|\delta u_k(jh)\| + \gamma_4^i \epsilon. \tag{44}$$

Thus, we can obtain that

$$\begin{aligned} \|\delta u_{k+1}(ih)\| &\leq \bar{\rho}_k(ih) \|\delta u_k(ih)\| + \gamma_6 \left(\gamma_5 \sum_{j=0}^{i-1} \gamma_4^{i-1-j} \|\delta u_k(jh)\| + \gamma_4^i \epsilon \right) \\ &\leq \bar{\rho}_k(ih) \|\delta u_k(ih)\| + \gamma_5\gamma_6 \sum_{j=0}^{i-1} \gamma_4^{i-1-j} \|\delta u_k(jh)\| + \gamma_6 \gamma_4^i \epsilon. \end{aligned} \tag{45}$$

Taking mathematical expectation to both sides of (45) yields

$$\begin{aligned} \mathbb{E}\, \|\delta u_{k+1}(ih)\| &\leq \mathbb{E}\, (\bar{\rho}_k(ih) \|\delta u_k(ih)\|) + \mathbb{E} \left(\gamma_5\gamma_6 \sum_{j=0}^{i-1} \gamma_4^{i-1-j} \|\delta u_k(jh)\| \right) + \gamma_6 \gamma_4^i \epsilon \\ &\leq \mathbb{E}\, (\bar{\rho}_k(ih))\, \mathbb{E}\, \|\delta u_k(ih)\| + \gamma_5\gamma_6 \sum_{j=0}^{i-1} \gamma_4^{i-1-j} \mathbb{E}\, \|\delta u_k(jh)\| + \gamma_6 \gamma_4^i \epsilon. \end{aligned} \tag{46}$$

Multiplying both sides of (46) with $\alpha^{-\lambda i}$ and taking supremum for all time instants i, we can get

$$\begin{aligned} \sup_i \alpha^{-\lambda i} \mathbb{E}\, \|\delta u_{k+1}(ih)\| &\leq \sup_i \mathbb{E}\, (\bar{\rho}_k(ih)) \sup_i \alpha^{-\lambda i} \mathbb{E}\, \|\delta u_k(ih)\| \\ &\quad + \gamma_5\gamma_6 \sup_i \alpha^{-\lambda i} \sum_{j=0}^{i-1} \gamma_4^{i-1-j} \mathbb{E}\, \|\delta u_k(jh)\| + \gamma_6 \epsilon \sup_i \alpha^{-(\lambda-1)i}. \end{aligned} \tag{47}$$

From (22), we obtain that

$$\|\delta u_{k+1}(ih)\|_\lambda \leq (\mu + \kappa\eta_d) \|\delta u_k(ih)\|_\lambda + \epsilon_1, \tag{48}$$

where $\epsilon_1 = \gamma_6\, \epsilon \sup_i \alpha^{-(\lambda-1)i}$. Then, it follows that

$$\lim_{k\to\infty} \|\delta u_k(ih)\|_\lambda \leq \frac{\epsilon_1}{1 - (\mu + \kappa\eta_d)}. \tag{49}$$

Moreover, from the relationship among $\delta u_k(ih)$, $\delta x_k(ih)$, and $e_k(ih)$, we have

$$\lim_{k\to\infty} \|e_k(ih)\|_\lambda \leq \frac{\epsilon_1 l_g \gamma_5 \eta_d}{1 - (\mu + \kappa\eta_d)} + l_g \sup_i \alpha^{-(\lambda-1)i} \epsilon. \tag{50}$$

It can be further obtained that

$$\lim_{k\to\infty} \sup_i \mathbb{E}\, \|e_k(ih)\| \leq \gamma_e \epsilon, \tag{51}$$

where $\gamma_e = \frac{\sup_i \alpha^i l_g \gamma_5 \gamma_6 \eta_d}{1-(\mu+\kappa\eta_d)} + l_g \sup_i \alpha^i$. This completes the proof. □

Generally, Theorem 3 shows that the initial state deviations linearly constrain the final tracking performance. Consequently, as $\epsilon \to 0$ (ie, the identically resetting condition holds), the tracking errors at sampling instants would converge to zero. This result coincides with our intuitive knowledge of the effect of initial states on the entire operation interval. In practical applications, we may design suitable initial learning mechanisms to achieve an asymptotically precise initialization.

Corollary 3. *Consider the continuous-time nonlinear system (1) with Assumptions 1-4 and 6. Let the PD-type ILC law (7) be applied. If the sampling period h is chosen small enough and the learning gains K_P and K_D satisfy that*

$$\begin{aligned} & 0 < (K_P + K_D)\psi_k(ih) < I, \\ & \sup_k \sup_i \| I - (K_P + K_D) p((i+1)h)\, \psi_k(ih) \| < 1, \end{aligned} \tag{52}$$

then the tracking error would converge to a small zone, whose upper bound is in proportion to ϵ, for all $i \in [1, n_d]$ as $k \to \infty$, ie, $\lim_{k\to\infty} \sup_i \mathbb{E}\|e_k(ih)\| \le \gamma_e{}'\epsilon$.

4.2 | The modified ILC scheme

Theorem 4. *Consider the continuous-time nonlinear system (1) with assumptions 1-4 and 6. Let the PD-type ILC law (29) be applied. If the sampling period h is chosen small enough and the learning gains $K_P{}'$ and $K_D{}'$ satisfy that*

$$0 \le \sum_{w=0}^{r} \vartheta_w \le \sigma < 1,$$

then the tracking error would converge to a small zone, whose upper bound is in proportion to ϵ, for all $i \in [1, n_d]$ as $k \to \infty$, ie, $\lim_{k\to\infty} \sup_i \mathbb{E}\|e_k(ih)\| \le \tilde{\gamma}_e \epsilon$.

Proof. From (34), we have

$$\|\delta x_{k-w}(ih)\| \le \gamma_5 \sum_{j=0}^{i-1} \gamma_4^{i-1-j} \|\delta u_{k-w}(jh)\| + \gamma_4^i \epsilon. \tag{53}$$

Substituting (53) into (33) implies that

$$\|\delta u_{k+1}(ih)\| \le \frac{1}{r+1} \sum_{w=0}^{r} \bar{\rho}_{k-w}{}'(ih) \|\delta u_{k-w}(ih)\| + \gamma_5\gamma_7 \cdot \frac{1}{r+1} \sum_{w=0}^{r} \sum_{j=0}^{i-1} \gamma_4^{i-1-j} \|\delta u_{k-w}(jh)\| + \frac{1}{r+1} \cdot \gamma_7 \gamma_4^i \epsilon. \tag{54}$$

Taking mathematical expectation on both sides of (54), we can obtain that

$$\begin{aligned} \mathbb{E}\|\delta u_{k+1}(ih)\| &\le \mathbb{E}\left(\frac{1}{r+1} \sum_{w=0}^{r} \bar{\rho}_{k-w}{}'(ih) \|\delta u_{k-w}(ih)\| \right) + \mathbb{E}\left(\gamma_5\gamma_7 \cdot \frac{1}{r+1} \sum_{w=0}^{r} \sum_{j=0}^{i-1} \gamma_4^{i-1-j} \|\delta u_{k-w}(jh)\| \right) + \frac{1}{r+1} \cdot \gamma_7 \gamma_4^i \epsilon \\ &\le \frac{1}{r+1} \sum_{w=0}^{r} \mathbb{E}\left(\bar{\rho}_{k-w}{}'(ih)\right) \mathbb{E}\|\delta u_{k-w}(ih)\| + \gamma_5\gamma_7 \cdot \frac{1}{r+1} \sum_{w=0}^{r} \sum_{j=0}^{i-1} \gamma_4^{i-1-j} \mathbb{E}\|\delta u_{k-w}(jh)\| + \frac{1}{r+1} \cdot \gamma_7 \gamma_4^i \epsilon. \end{aligned} \tag{55}$$

Multiplying both sides of (55) with $\alpha^{-\lambda i}$ and taking supremum for all time instants i, we can get

$$\begin{aligned} \sup_i \alpha^{-\lambda i} \mathbb{E}\|\delta u_{k+1}(ih)\| \le{}& \frac{1}{r+1} \sum_{w=0}^{r} \sup_i \mathbb{E}\left(\bar{\rho}_{k-w}{}'(ih)\right) \sup_i \alpha^{-\lambda i} \mathbb{E}\|\delta u_{k-w}(ih)\| \\ & + \gamma_5\gamma_7 \cdot \frac{1}{r+1} \sup_i \alpha^{-\lambda i} \sum_{w=0}^{r} \sum_{j=0}^{i-1} \gamma_4^{i-1-j} \mathbb{E}\|\delta u_{k-w}(jh)\| + \gamma_7 \epsilon \cdot \frac{1}{r+1} \sup_i \alpha^{-(\lambda-1)i}. \end{aligned} \tag{56}$$

Then,

$$\|\delta u_{k+1}(ih)\|_\lambda \le \frac{1}{r+1} \sum_{w=0}^{r} \sup_i \mathbb{E}\left(\bar{\rho}_{k-w}{}'(ih)\right) \|\delta u_{k-w}(ih)\|_\lambda + \gamma_5\gamma_7 \cdot \frac{1}{r+1} \eta_d \sum_{w=0}^{r} \|\delta u_{k-w}(ih)\|_\lambda + \gamma_7 \epsilon \cdot \frac{1}{r+1} \sup_i \alpha^{-(\lambda-1)i}. \tag{57}$$

Therefore,

$$\|\delta u_{k+1}(ih)\|_\lambda \le \left(\sum_{w=0}^{r} \vartheta_w + \theta \right) \max\left\{ \|\delta u_k(ih)\|_\lambda, \|\delta u_{k-1}(ih)\|_\lambda, \ldots, \|\delta u_{k-r+1}(ih)\|_\lambda \right\} + \epsilon_2, \tag{58}$$

where $\epsilon_2 = \gamma_7\epsilon \cdot \frac{1}{r+1}\sup_i \alpha^{-(\lambda-1)i}$. It follows that

$$\lim_{k\to\infty} \|\delta u_k(ih)\|_\lambda \le \frac{\epsilon_2}{1-\left(\sum_{w=0}^{r}\vartheta_w+\theta\right)}. \tag{59}$$

Moreover, from the relationship among $\delta u_k(ih)$, $\delta x_k(ih)$, and $e_k(ih)$, we have

$$\lim_{k\to\infty} \|e_k(ih)\|_\lambda \le \frac{\epsilon_2 l_g\gamma_5\eta_d}{1-\left(\sum_{w=0}^{r}\vartheta_w+\theta\right)} + l_g\epsilon\sup_i \alpha^{-(\lambda-1)i}. \tag{60}$$

It can be further obtained that

$$\lim_{k\to\infty}\sup_i \mathbb{E}\,\|e_k(ih)\| \le \tilde{\gamma}_e\epsilon, \tag{61}$$

where $\tilde{\gamma}_e = \frac{1}{r+1}\cdot\frac{\sup_i\alpha^i l_g\gamma_5\gamma_7\eta_d}{1-(\sum_{w=0}^{r}\vartheta_w+\theta)} + l_g\sup_i\alpha^i$. This completes the proof. □

Similar to Theorem 2, this theorem extends previous results to the iteration-moving-average-operator–based ILC algorithm and provides the sufficient condition for convergence. The dependence of the final tracking error on the initial state error is also described.

Corollary 4. *Consider the continuous-time nonlinear system (1) with Assumptions 1-4 and 6. Let the PD-type ILC law (29) be applied. If the sampling period h is chosen small enough and the learning gains $K_P{}'$ and $K_D{}'$ satisfy that*

$$\begin{aligned} &0 < \left(K_P{}' + K_D{}'\right)\psi_{k-w}(ih) < I, \\ &0 \le \sum_{w=0}^{r}\vartheta_w{}' < 1, \end{aligned} \tag{62}$$

then the tracking error would converge to a small zone, whose upper bound is in proportion to ϵ, for all $i \in [1, n_d]$ as $k \to \infty$, ie, $\lim_{k\to\infty}\sup_i \mathbb{E}\|e_k(ih)\| \le \tilde{\gamma}_e{}'\epsilon$.

5 | NUMERICAL EXPERIMENTS

In this section, an illustration example is presented to show the effectiveness of 2 proposed ILC schemes.

Consider the following continuous-time nonlinear system:

$$\begin{aligned} \dot{x}_1(t) &= 1.8x_2(t), \\ \dot{x}_2(t) &= 2.2\cos(x_2(t)) + 2.2u(t) + 0.1\sin(x_1(t)u(t)), \\ y(t) &= x_1(t). \end{aligned} \tag{63}$$

The length of desired trajectory is $T_d = 1$. In order to simulate the randomly iteration-varying length, let the actual length T_k vary among $[0.9, 1]$. We choose $h = 0.005$ as the sampling period, then the expected sampling number is $n_d = 200$. Thus, n_k varies among $[180, 200]$. For a simple simulation, we assume that n_k obeys the uniform distribution on $[180, 200]$.

The desired reference trajectory is

$$y_d(ih) = 3\pi(ih)^3 - \frac{1}{7}\pi(ih)^7.$$

5.1 | Generic PD-type ILC scheme

The learning law (7) is applied with the learning gains given as $K_P = 0.1$ and $K_D = 190$. It is numerically computed that the condition (8) is satisfied. The initial state for each iteration is first set to be $x_k(0) = 0$ (according to Assumption 5). The algorithm runs for 100 iterations. Figure 1 shows that the output trajectory converges to the desired trajectory at all

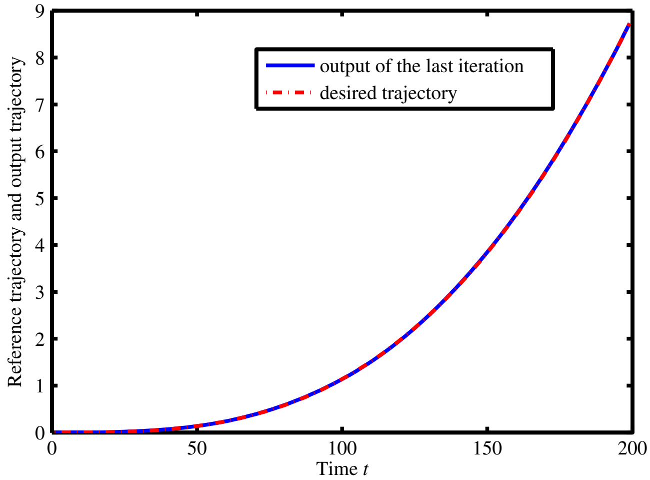

FIGURE 1 Desired trajectory and output of the last iteration [Colour figure can be viewed at wileyonlinelibrary.com]

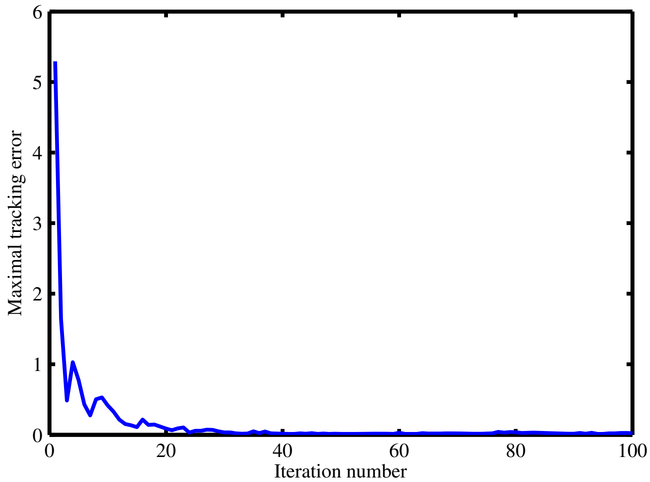

FIGURE 2 Maximal tracking error along iterations [Colour figure can be viewed at wileyonlinelibrary.com]

sampling instants for the last iteration, ie, the 100th iteration. It is seen that the output at the last iteration almost coincides with the desired reference, which shows the well tracking performance of the generic PD-type ILC.

The performance of the maximal tracking error is presented in Figure 2, where the maximal tracking error is defined as the worst tracking error of each iteration. We can observe from Figure 2 that the maximal tracking error decreases fast at the first few iterations and then converges to zero asymptotically along the iteration axis. Moreover, to show the tracking performance and the iteration-varying lengths, we plot the tracking error of the whole iteration in Figure 3, where the 60th, 70th, and 80th iterations are illustrated, respectively. As one can see, the magnitude of the tracking error is rather small for the referred iterations. Meanwhile, the lengths of the 60th, 70th, and 80th iterations are 195, 191, and 196, respectively. This observation demonstrates the fact that the iteration length can vary from iteration to iteration. In addition, we also find that the tracking error of the latter part of the time interval is distinctly larger than that of the former part of the time interval. This is because the tracking error from previous time instants will also affect the tracking error at the latter time instants. Thus, it is reasonable that the tracking error from the former part would converge faster than that from the latter part.

In addition, to show the robustness of the proposed algorithm against randomly varying initial states, we let ϵ in Assumption 6 be 0.02, 0.06, and 0.12, respectively. Then, the algorithm still runs for 100 iteration according to each case and the maximal tracking error profiles along the iteration axis are plotted in Figure 4. It can be seen that the upper bounds of the maximal tracking errors are strongly related to the value of ϵ; that is, a larger ϵ leads to a larger bound of maximal tracking errors profiles. This verifies the theoretical analysis.

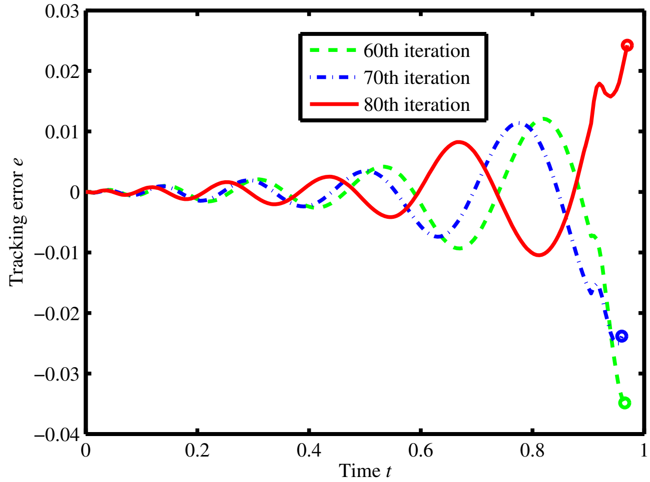

FIGURE 3 Tracking errors at the 60th, 70th, and 80th iterations [Colour figure can be viewed at wileyonlinelibrary.com]

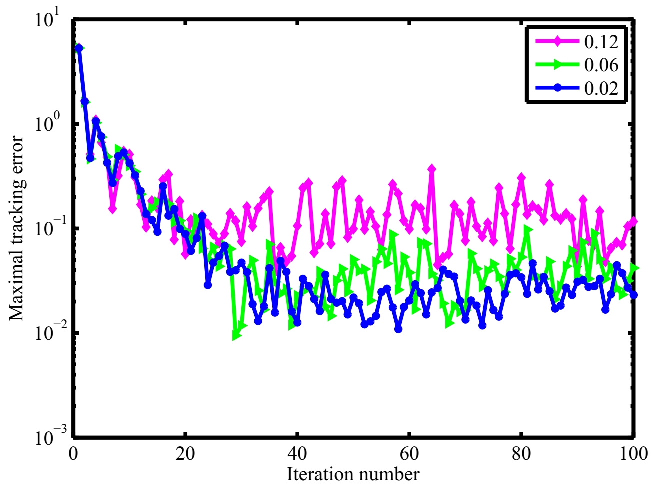

FIGURE 4 Maximal tracking errors along iterations [Colour figure can be viewed at wileyonlinelibrary.com]

5.2 | The modified ILC scheme

The modified ILC scheme (29) is also simulated for 100 iterations. The parameters of the learning algorithm are same to the generic ILC (7), except that there exists an averaging operator with the moving window size being four. That is, in the modified ILC scheme, we still retain the learning gains as $K_P = 0.1$ and $K_D = 190$ and let $r = 3$. Thus, the signals from the kth, $(k-1)$th, $(k-2)$th, and $(k-3)$th iterations are used in generating the input signal for the $(k+1)$th iteration.

We first simulate the identical initialization case, ie, $x_k(0) = 0$ (according to Assumption 5). The output tracking performance of the last iteration is the same to that of Figure 1, and thus we omit this Figure to avoid repetition. The maximal tracking error profile along the iteration axis and the illustrated tracking error profiles along the time axis for selected iterations are plotted in Figures 5 and 6, respectively. Then, we simulate the varying initial states case following the same setting in the last subsection. That is, we let ϵ in Assumption 6 be 0.02, 0.06, and 0.12, respectively. The maximal tracking error profiles along the iteration axis are plotted in Figure 7. It is observed that the conclusions for the generic ILC scheme still hold for the modified ILC scheme.

Some interesting observations are noted by comparing the related figures for the generic ILC algorithm and the modified one. First of all, both of them are effective for achieving the precise tracking performance with sampled-data as shown in Figure 1. This demonstrates the effectiveness of both algorithms. Moreover, comparing Figures 2 and 5, we find that the convergence speed of the modified scheme is a little slower than the generic scheme. This is because the generic algorithm (7) is more sensitive to the latest information as it only use the information from the last iteration for its updating, while the modified algorithm (29) would make an average to the information coming from adjacent iterations.

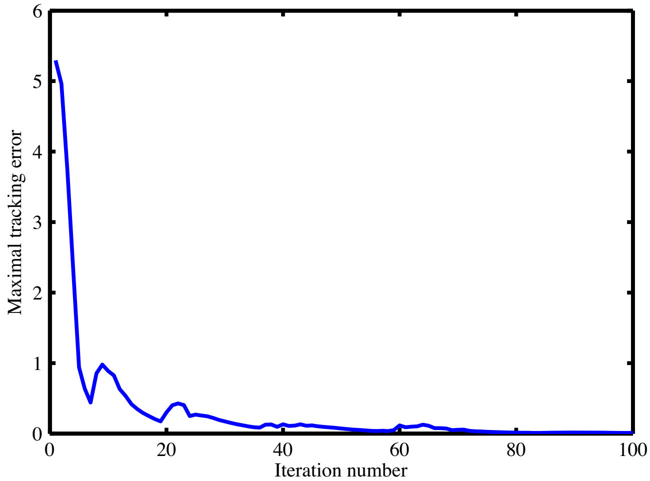

FIGURE 5 Maximal tracking error along iterations [Colour figure can be viewed at wileyonlinelibrary.com]

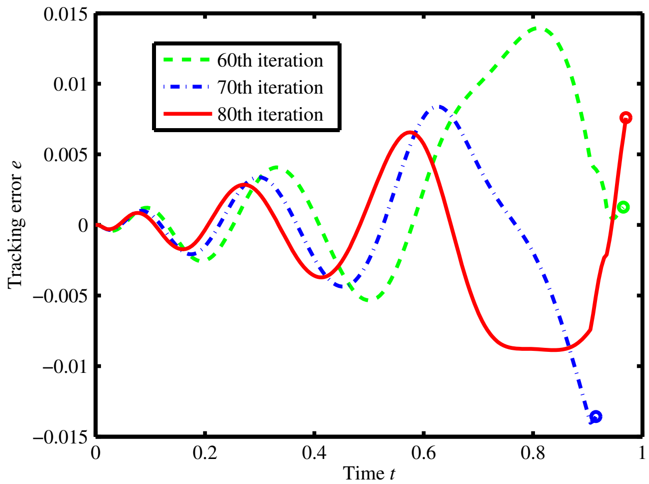

FIGURE 6 Tracking errors at the 60th, 70th, and 80th iterations [Colour figure can be viewed at wileyonlinelibrary.com]

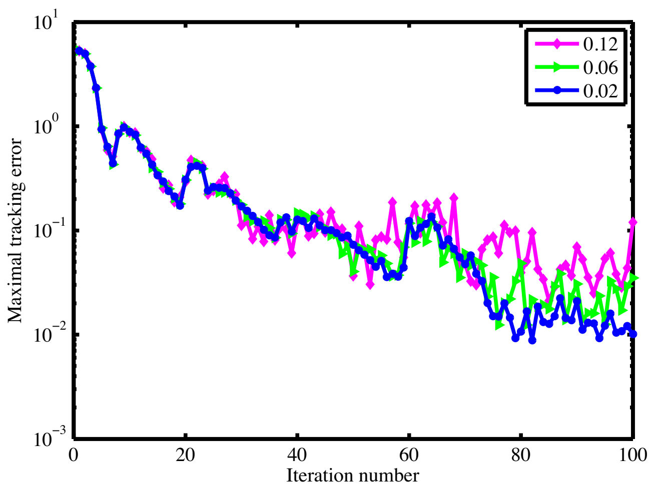

FIGURE 7 Maximal tracking errors along iterations [Colour figure can be viewed at wileyonlinelibrary.com]

However, as shown in Figures 3 and 6, within the same iterations and for the varying time interval (ie, $180 \leq n_k \leq 200$), the magnitude of the tracking error profiles of the modified algorithm (29) is generally smaller than that of the generic algorithm (7). The reason lies in the fact that the average mechanism in the modified algorithm would bring us robustness against the varying length problem, which makes a successive improvement of the tracking performance along the iteration axis. On the other hand, without such mechanism, the generic algorithm is more possible to be affected a lot when encountering bad situations. Similar performance also exists in the varying initial state case, as shown in Figures 4 and 7, where the modified algorithm provides a more attractive improvement of the tracking performance than the generic algorithm.

In addition, one may interest in how is the effect of the moving window size to the tracking performance for the modified algorithm (29). This is an important issue for further study. Generally, the design of the moving window size, ie, $r + 1$, depends on the system dynamics, the nonlinearity of the system, the varying length range, the distribution of the varying length, among other factors. Thus, it is a hard work to give an explicit expression of the moving size $r + 1$ according to the system information and process environments. However, we may give some general guidelines for the selection of the moving window size for practical applications. First, we usually select the size from three to five. The algorithm would behave less well when the size is too small or too large. Moreover, if the random interval of n_k is long, we usually select a large size because this case implies that the iteration length varies drastically in the iteration domain and more previous information is required to make up the missing data. Otherwise, if the random interval of n_k is short, a small size is preferable to avoid redundancy of historical information. In short, the selection of the moving window size is a trade-off among various factors.

6 | CONCLUSION

In this paper, the sampled-data ILC problem for continuous-time nonlinear systems with iteration-varying lengths and higher relative degree is discussed. To achieve the control objective, 2 sampled-data ILC schemes are proposed with modified tracking errors, namely, the generic PD-type ILC scheme and the PD-type ILC algorithm incorporated with an iteratively moving average operator. Moreover, the probability distribution of the random trial length is not required prior in this paper. For the identical initial state case, if the sampling period is set to be small enough and certain conditions are satisfied for the learning gains, the system output at each sampling instant has been shown to converge to the desired trajectory as the iteration number goes to infinity for both algorithms. For the varying initial state case, both algorithms are also effective in the sense that the tracking errors converge to a small zone with its upper bound being in proportion to the initial state magnitude. For further research, it is of great interest to make a deep investigation on the relationship between the moving window size and the operation environments.

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