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Learning formation control for fractional-order multiagent systems

Dahui Luo[1] | JinRong Wang[1,2] | Dong Shen[3]

[1] Department of Mathematics, Guizhou University, Guiyang 550025, China

[2] School of Mathematical Sciences, Qufu Normal University, Qufu 273165, China

[3] College of Information Science and Technology, Beijing University of Chemical Technology, Beijing 100029, China

Correspondence
JinRong Wang, Department of Mathematics, Guizhou University, Guiyang 550025, China.
Email: jrwang@gzu.edu.cn

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In this paper, we use 2 iterative learning control schemes (*P*-type and *PI*-type) with an initial learning rule to achieve the formation control of linear fractional-order multiagent systems. To realize the finite-time consensus, we assume repeatable operation environments as well as a fixed but directed communication topology for the fractional-order multiagent systems. Both *P*-type and *PI*-type update laws are applied to generate the control commands for each agent. It is strictly proved that all agents are driven to achieve an asymptotical consensus as the iteration number increases. Two examples are simulated to verify the effectiveness of the proposed algorithms.

KEYWORDS
convergence analysis, fractional order, iterative learning control, multiagent systems

1 | INTRODUCTION

Fractional-order calculus has a long history over 300 years.[1,2] Recently, systems with different fractional-order derivatives are used to characterize some certain evolution process in viscoelasticity control.[3,4] As fractional-order systems have distinctly different evolution properties from ordinary systems,[5,6] it is desired to consider the specific control techniques for fractional-order systems. In this paper, we are interested in iterative learning control (ILC), which was first proposed by Arimoto in 1980s for tracking problem in robotics.[7] After developments of over 3 decades, it had been wildly used to deal with process control of artificial intelligence systems.[8,9] However, most existing literature concentrates on ordinary system, while only very a few papers[10-14] consider ILC for fractional-order systems, showing that much blank exists for further attention.

Multiagent systems (MAS) have been found considerable applications in cross-disciplinary nature.[15-18] Although there are certain early contributions reported on MAS coordination problem with each agent described by a fractional-order model,[19-21] it is still on the initial stage of the topic. Many issues are still open for achieving better consensus/coordination performance. In fact, the topic of learning in MAS has been one of the most fertile grounds for interaction between game theory and artificial intelligence.[22] Therefore, it motivates us to introduce certain learning idea to improve the formation tracking performance of a factional-order MAS (FOMAS). A pioneer monograph[23] and some research papers[24-28] have successfully used ILC rules to solve coordination/formation control problems for ordinary MAS, and the research of ILC for FOMAS is still blank.

Clearly, the FOMAS possesses certain interleaving effect among different agents compared with traditional MAS; that is, the tracking performance of previous time interval will be involved with certain memory property for all agents. In this paper, we use 2 ILC schemes (*P*-type and *PI*-type) with an initial learning rule to achieve the formation control of linear FOMAS, where the topology of agents is described by a fixed but directed graph. The finite-time formation is asymptotically achieved as the iteration number increases. To our best knowledge, this paper is the first result on ILC for FOMAS.

The rest of this paper is organized as follows. In Section 2, we use graph theory to formulate the consensus tracking problem of FOMAS. In Section 3, we provide strict convergence analysis of both *P*-type and *PI*-type ILC schemes with an initial state learning rule. Simulation examples are demonstrated in final section to verify the theoretical results.

2 | PRELIMINARIES AND PROBLEM FORMULATION

We collect some knowledge of graph theory to formulate MAS (see Yang et al[23] for details). Let $\Omega=(V,E,A)$ be a weighted directed graph, $V=\{1,2,\cdots,N\}$ be the set of vertices, $E\subseteq V\times V$ be the set of edges, and A be the adjacency matrix. Here, we understand that V denotes the index set representing the agents in the MAS. We write a pair $(i,j)\in E$ as a direct edge from i to j, that is, agent j can receive information from agent i. A path between vertices p and q is a sequence $(p=j_1,\cdots,j_l=q)$ of distinct vertices such that all pairs $(j_k,j_{k+1})\in E$, $\forall 1\leq k\leq l-1$. That is, a path is a combination of successive pairs. Then, we say i is the parent of j, and j is the child of i. The set of neighbors of ith agent is denoted by $N_i=\{j\in V:(j,i)\in E\}$. $A=(a_{ij})\in R^{n\times n}$ is the weighted adjacency matrix of G with $a_{ij}\geq 0$. In particular, $a_{ij}=1$ if $(j,i)\in E$ and $i\neq j$, and 0 otherwise. Denote $d_i^{in}=\sum_{j=1}^{N}a_{i,j}$ be the in-degree of vertex i, $D=diag(d_1^{in},\ \cdots,d_N^{in})$, and $L=D-A$ be the Laplacian of G. The concept of a spanning tree is a directed graph, whose vertices have exactly one parent except for one vertex; in this case, the root has no parent. Once V and a subset of E can formulate a spanning tree, we call that graph has a spanning tree.

Denote $\otimes$ by Kronecker product. Set $A=(a_{ij})\in R^{m\times n}$, $B\in R^{p\times q}$, and $A\otimes B=\begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}\in R^{mp\times nq}$. For some matrices A, B, C, and D of appropriate dimensions, $k(A\otimes B)=A\otimes kB$, $(A+B)\otimes C=A\otimes C+B\otimes C$, $(A\otimes B)(C\otimes D)=AC\otimes BD$, and $\|A\otimes B\|=\|A\|\cdot\|B\|$.

Throughout the paper, denote both vector norm and its compatible matrix norm by $\|\cdot\|$. The standard λ-norm for a function $g:[0,T]\to R^n$ is defined as $\|g\|_\lambda=\sup_{t\in[0,T]}e^{-\lambda t}\|g(t)\|$ where $\lambda>0$.

Consider a group of N fractional-order agents and their interaction topology can be described by $\Omega=(V,E,A)$. The ith agent is governed by the following linear fractional-order model:

$$ {}_0^cD_t^\alpha x_i(t)=Ax_i+Bu_i, \tag{1}$$

$$ y_i=Cx_i+Du_i, \tag{2}$$

$\alpha\in(0,1)$, $t\in[0,T]$, $i\in V$, where ${}_0^cD_t^\alpha x_i(t):=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{x_i'(s)}{(t-s)^\alpha}ds$ denotes the Caputo fractional derivative of order α for x_i with the lower limit zero (see Kilbas et al[1]), $x_i\in R^{n\times 1}$ is the state vector, $y_i\in R^{m\times 1}$ is the output vector, $u_i\in R^{m\times 1}$ is the control input, and $A\in R^{n\times n}$, $B\in R^{n\times m}$, $C\in R^{m\times n}$, and $D\in R^{m\times m}$ are the constant matrices with $rank(CB)=m$.

Let $y_d(t),t\in[0,T]$ be the desired sufficiently smooth trajectory for consensus tracking, which is accessible to a subset of followers only. Next, we regard $y_d(t),t\in[0,T]$ as a leader and index it by vertex 0 in the graph. Then, the united graph describing the information flow among the leader and its followers can be expressed as $\Omega^+=(V\cup\{0\},E^+,A^+)$, where E^+ is the edge set and A^+ is the weighted adjacency matrix of Ω^+.

Throughout the paper, we assume that the desired trajectory y_d is realizable for all agents, and there exist x_d and u_d such that $y_d=Cx_d+Du_d$. Our control objective is to design appropriate learning schemes to guarantee that the outputs from all agents asymptotically achieve to the desired trajectory over a finite time interval.

3 | MAIN RESULTS

Let $\eta_{k,j}(t)$ be the available information at the $(k+1)$th iteration for the jth agent, where k denotes the iteration number and j denotes the agent index. Consider

$$\eta_{k,j} = \sum_{\omega \in N_j} a_{j,\omega}(y_{k,\omega}(t) - y_{k,j}(t)) + s_j(y_d(t) - y_{k,j}(t)), \tag{3}$$

where s_j equals 1 if the jth agent can access the desired trajectory and 0 otherwise.

For (1) and (2), we consider the P-type and PI-type ILC updating rules with the initial state learning rules, respectively: $u_{k+1,j}(t)=u_{k,j}(t)+\varphi\eta_{k,j}(t)$ and $x_{k+1,j}(0)=x_{k,j}(0)+\psi\eta_{k,j}(0)$, and $u_{k+1,j}(t) = u_{k,j}(t) + \varphi\eta_{k,j}(t) + \omega\int_0^t \eta_{k,j}(\tau)d\tau$ and $x_{k+1,j}(0)=x_{k,j}(0)+\psi\eta_{k,j}(0)$, where $\varphi \in R^{m\times m}$, $\psi \in R^{n\times m}$, and $\omega \in R^{n\times m}$ are constant learning gain matrices.

Let $e_{k,j}(t)=y_d(t)-y_{k,j}(t)$ be the tracking error. Then we rewrite (3) as

$$\eta_{k,j} = \sum_{\omega \in N_j} a_{j,\omega}(e_{k,j}(t) - e_{k,\omega}(t)) + s_j e_{k,j}(t), \tag{4}$$

in the terms of error.

For the kth iteration, we define the column stack vectors: $\eta_k(t)=[\eta_{k,1}(t)^T,\eta_{k,2}(t)^T,\cdots,\eta_{k,N}(t)^T]^T$, $x_k(t)=[x_{k,1}(t)^T,x_{k,2}(t)^T,\cdots,x_{k,N}(t)^T]^T$, $e_k(t)=[e_{k,1}(t)^T,e_{k,2}(t)^T,\cdots,e_{k,N}(t)^T]^T$. Thus, linking (4) and both P-type and PI-type ILC laws via Kronecker product, we obtain $\eta_k = ((L+S)\otimes I_m)e_k(t)$,

$$u_{k+1}(t) = u_k(t) + ((L+S)\otimes\varphi)e_k(t), \tag{5}$$

$$x_{k+1}(0) = x_k(0) + ((L+S)\otimes\psi)e_k(0), \tag{6}$$

and

$$u_{k+1}(t) = u_k(t) + ((L+S)\otimes\varphi)e_k(t) + ((L+S)\otimes\omega)\int_0^t e_k(\tau)d\tau, \tag{7}$$

$$x_{k+1}(0) = x_k(0) + ((L+S)\otimes\psi)e_k(0), \tag{8}$$

where I_m and L denote $m\times m$ identity matrix and graph Laplacian of Ω, respectively, and $S=diag(s_1,s_2,\cdots,s_N)$, $s_i\geq 0$, $i=1,2,\cdots,N$ is associated with Ω^+; $\varphi \in R^{m\times m}$, $\psi \in R^{n\times m}$, and $\omega \in R^{n\times m}$ are constant learning gain matrices.

3.1 | Convergence analysis of *P*-type ILC for (1) and (2)

Theorem 3.1. *For (1) and (2), considering (5) and (6), the consensus tracking error* $e_k(t)\to 0$ *as iteration* $k\to\infty$, *ie,* $\lim_{k\to\infty} y_{k,j}(t) = y_d(t)$ *for all* $t\in[0,T]$ *provided that the virtual leader has a path to any follower agent and*

$$\| I_{mN} - (L+S)\otimes C\psi - (L+S)\otimes D\varphi \| < 1, \tag{9}$$

$$\| I_{mN} - (L+S)\otimes D\varphi \| < 1. \tag{10}$$

Proof. We calculate that

$$\begin{aligned} e_{k+1}(0) &= y_d(0) - y_{k+1}(0) \\ &= e_k(0) - \left(y_{k+1}(0) - y_k(0)\right) \\ &= e_k(0) - ((I_N\otimes C)(x_{k+1}(0) - x_k(0)) + (I_N\otimes D)(u_{k+1}(0) - u_k(0))) \\ &= e_k(0) - (((L+S)\otimes C\psi)e_k(0) + ((L+S)\otimes D\varphi)e_k(0)) \\ &= (I_{mN} - (L+S)\otimes C\psi - (L+S)\otimes D\varphi)\, e_k(0), \end{aligned}$$

which yields that

$$\| e_{k+1}(0) \| \leq \| I_{mN} - (L+S)\otimes C\psi - (L+S)\otimes D\varphi \| \| e_k(0) \|.$$

Since the leader has a path to any follower agent, then $L+S$ is nonsingular and all the eigenvalues have positive real parts. This assumption then guarantees the possible existence of ψ and φ for ensuring $\|I_{mN}-(L+S)\otimes C\psi-(L+S)\otimes D\varphi\|<1$. Then, by (9), we have

$$\lim_{k\to\infty}\|e_k(0)\|\to 0. \tag{11}$$

Next, applying (5) and (6) to all the agents, we have

$$\begin{aligned}
x_{k+1}(t) &= x_{k+1}(0) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(I_N\otimes A)x_{k+1} + (I_N\otimes B)u_{k+1}}{(t-\tau)^{1-\alpha}}d\tau \\
&= x_k(0) + ((L+S)\otimes\psi)e_k(0) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(I_N\otimes A)x_{k+1} + (I_N\otimes B)u_{k+1}}{(t-\tau)^{1-\alpha}}d\tau \\
&= x_k(t) + ((L+S)\otimes\psi)\ \ e_k(0) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(I_N\otimes A)(x_{k+1}(\tau)-x_k(\tau))}{(t-\tau)^{1-\alpha}}\,d\tau \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{((L+S)\otimes B\varphi)e_k(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau.
\end{aligned}$$

This gives that

$$\begin{aligned}
\|\Delta x_k(t)\| &= \|x_{k+1}(t)-x_k(t)\| \\
&\le \|L+S\|\|\psi\|\|e_k(0)\| + \frac{\|A\|}{\Gamma(\alpha)}\int_0^t \frac{\|\Delta x_k(\tau)\|}{(t-\tau)^{1-\alpha}}\,d\tau \\
&\quad + \frac{\|L+S\|\|B\|\|\varphi\|}{\Gamma(\alpha)}\int_0^t \frac{\|e_k(\tau)\|}{(t-\tau)^{1-\alpha}}\,d\tau.
\end{aligned} \tag{12}$$

Next, multiplying the factor of $e^{-\lambda t}$ on both sides of (12),

$$\begin{aligned}
\|\Delta x_k(t)\|e^{-\lambda t} &\le \|L+S\|\|\psi\|\|e_k(0)\|e^{-\lambda t} + \frac{\|A\|}{\Gamma(\alpha)}\,e^{-\lambda t}\int_0^t \frac{\|\Delta x_k(\tau)\|e^{-\lambda\tau}e^{\lambda\tau}}{(t-\tau)^{1-\alpha}}\,d\tau \\
&\quad + \frac{\|L+S\|\|B\|\|\varphi\|}{\Gamma(\alpha)}\,e^{-\lambda t}\int_0^t \frac{\|e_k(\tau)\|e^{-\lambda\tau}e^{\lambda\tau}}{(t-\tau)^{1-\alpha}}\,d\tau. \\
&\le \|L+S\|\|\psi\|\|e_k(0)\|e^{-\lambda t} + \frac{\|A\|}{\Gamma(\alpha)}\,e^{-\lambda t}\int_0^t (t-\tau)^{\alpha-1}e^{\lambda\tau}d\tau\|\Delta x_k\|_\lambda \\
&\quad + \frac{\|L+S\|\|B\|\|\varphi\|}{\Gamma(\alpha)}\,e^{-\lambda t}\int_0^t (t-\tau)^{1-\alpha}e^{\lambda\tau}d\tau\|e_k\|_\lambda.
\end{aligned} \tag{13}$$

Using Hölder inequality, one has

$$\begin{aligned}
\int_0^t (t-\tau)^{\alpha-1}e^{\lambda\tau}d\tau &\le \left(\int_0^t (t-\tau)^{p(\alpha-1)}d\tau\right)^{\frac{1}{p}}\left(\int_0^t e^{q\lambda\tau}d\tau\right)^{\frac{1}{q}} \\
&= \sqrt[p]{\frac{1}{1-p(1-\alpha)}}\,t^{\frac{1}{p}-(1-\alpha)}\,\frac{1}{\sqrt[q]{q\lambda}}\sqrt[q]{e^{q\lambda t}-1} \\
&\le \sqrt[p]{\frac{1}{1-p(1-\alpha)}}\,t^{\frac{1}{p}-(1-\alpha)}\,\frac{1}{\sqrt[q]{q\lambda}e^{\lambda t}},
\end{aligned} \tag{14}$$

where $\frac{1}{p}+\frac{1}{q}=1$ *and* $p\in(1,\frac{1}{1-\alpha}),\ q>0.$

Submitting (14) into (13) and taking supremum, we have

$$\begin{aligned}
\|\Delta x_k\|_\lambda \ \le\ & \|L+S\|\|\psi\|\|e_k(0)\| + \frac{\|A\|}{\Gamma(\alpha)}\|\Delta x_k\|_\lambda\sqrt[p]{\frac{1}{1-p(1-\alpha)}}T^{\frac{1}{p}-(1-\alpha)}\frac{1}{\sqrt[q]{q\lambda}} \\
&+ \frac{\|L+S\|\|B\|\|\varphi\|}{\Gamma(\alpha)}\sqrt[p]{\frac{1}{1-p(1-\alpha)}}T^{\frac{1}{p}-(1-\alpha)}\frac{1}{\sqrt[q]{q\lambda}\|e_k\|_\lambda}.
\end{aligned}$$

Obviously, for some λ large enough, we have

$$\|\Delta x_k\|_\lambda \le \|L+S\|\|\psi\|\|e_k(0)\|. \tag{15}$$

Note that

$$\begin{aligned} e_{k+1}(t) &= e_k(t)-((I_N\otimes C)(x_{k+1}(t)-x_k(t))+(I_N\otimes D)(u_{k+1}(t)-u_k(t))) \\ &= (I_{mN}-(L+S)\otimes D\varphi)\, e_k(t)-(I_N\otimes C)\,(x_{k+1}(t)-x_k(t)) \end{aligned}.$$

This yields that

$$\|e_{k+1}(t)\| \le \|I_{mN}-(L+S)\otimes D\varphi\|\|e_k(t)\|+\|C\|\|x_{k+1}(t)-x_k(t)\|. \tag{16}$$

Taking λ-norm for both side of (16), we have

$$\|e_{k+1}\|_\lambda \le \|I_{mN}-(L+S)\otimes D\varphi\|\|e_k\|_\lambda+\|C\|\|\Delta x_k\|_\lambda. \tag{17}$$

Submitting (15) into (17), we have

$$\|e_{k+1}\|_\lambda \le \|I_{mN}-(L+S)\otimes D\varphi\|\|e_k\|_\lambda+\|C\|\|L+S\|\|\psi\|\|e_k(0)\|.$$

Then, we have

$$\begin{aligned} \|e_{k+1}\|_\lambda &\le \|I_{mN}-(L+S)\otimes D\varphi\|^{k+1-n}\|e_n\|_\lambda \\ &+\sup_{n\le l\le k}\frac{\|C\|\|L+S\|\|\psi\|\|e_l(0)\|}{1-\|I_{mN}-(L+S)\otimes D\varphi\|}\left(1-\|I_{mN}-(L+S)\otimes D\varphi\|^{k+1-n}\right), \end{aligned}$$

for some $k>n\ge 1$ where n is an arbitrary integer. This further implies that

$$\lim_{k\to\infty}\|e_{k+1}\|_\lambda \le \sup_{l\ge n}\frac{\|C\|\|L+S\|\|\psi\|\|e_l(0)\|}{1-\|I_{mN}-(L+S)\otimes D\varphi\|},\quad \forall n\ge 1.$$

Thus, by the arbitrariness of n, we have

$$\lim_{k\to\infty}\sup\|e_k\|_\lambda \le \frac{\|C\|\|L+S\|\|\psi\|\lim_{k\to\infty}\|e_k(0)\|}{1-\|I_{mN}-(L+S)\otimes D\varphi\|}.$$

Again, the assumption that the virtual has a path to any follower ensures that $L+S$ is nonsingular (otherwise, $\|I_{mN}-(L+S)\otimes D\varphi\|=1$). By (11), the denominator is nonzero and we have $\lim_{k\to\infty}\|e_k\|_\lambda=0$. The proof is finished.

Remark 3.2. In the theorem, we assume the connection topology by requiring that the virtual leader has a path to any follower agent possibly passing through several other agents. In other words, the directed communication graph including the virtual leader and all agents/followers is assumed to contain a spanning tree with the virtual leader being the root. Such assumption is a necessary communication requirement for solvability of the consensus tracking problem. If there is an isolated agent (ie, there is none path from the virtual leader to this agent), it is impossible for that agent to follow the leaders trajectory as it does not even know the control objective.

3.2 | *PI*-type ILC for (1) and (2)

Theorem 3.3. *For (1) and (2), considering (7) and (8), the consensus tracking error* $e_k(t)\to 0$ *as iteration* $k\to\infty$, *ie,* $\lim_{i\to\infty}y_{k,j}(t)=y_d(t)$ *for all* $t\in[0,T]$ *provided that the virtual leader has a directed path to any follower agent and (9) and (10) hold.*

Proof. Applying (7) and (8) to all the agents, we have

$$\begin{aligned}\Delta x_k(t) &= ((L+S)\otimes\psi)\ e_k(0)+\frac{I_N\otimes A}{\Gamma(\alpha)}\int_0^t\frac{\Delta x_k(\tau)}{(t-\tau)^{1-\alpha}}\ d\tau\\ &\quad+\frac{(L+S)\otimes B\varphi}{\Gamma(\alpha)}\int_0^t\frac{e_k(\tau)}{(t-\tau)^{1-\alpha}}\ d\tau+\frac{(L+S)\otimes B\omega}{\Gamma(\alpha)}\int_0^t\frac{\int_0^\tau e_k(s)ds}{(t-\tau)^{1-\alpha}}\ d\tau.\end{aligned}$$

This yields that

$$\begin{aligned}\|\Delta x_k(t)\| &\le \|L+S\|\|\psi\|\|e_k(0)\|+\frac{\|A\|}{\Gamma(\alpha)}\int_0^t\frac{\|\Delta x_k(\tau)\|}{(t-\tau)^{1-\alpha}}d\tau\\ &\quad+\frac{\|L+S\|\|B\|\|\varphi\|}{\Gamma(\alpha)}\int_0^t\frac{\|e_k(\tau)\|}{(t-\tau)^{1-\alpha}}d\tau+\frac{\|L+S\|\|B\|\|\omega\|}{\Gamma(\alpha)}\int_0^t\frac{\int_0^\tau\|e_k(s)\|ds}{(t-\tau)^{1-\alpha}}d\tau.\end{aligned}$$

Further, one can deduce

$$\begin{aligned}\|\Delta x_k(t)\|e^{-\lambda t} &\le \|L+S\|\|\psi\|\|e_k(0)\|e^{-\lambda t}+\frac{\|A\|}{\Gamma(\alpha)}\sqrt[p]{\frac{1}{1-p(1-\alpha)}}\ t^{\frac{1}{p}-(1-\alpha)}\frac{1}{\sqrt[q]{q\lambda}\|\Delta x_k\|_\lambda}\\ &\quad+\frac{\|L+S\|\|B\|\|\varphi\|}{\Gamma(\alpha)}\sqrt[p]{\frac{1}{1-p(1-\alpha)}}\ t^{\frac{1}{p}-(1-\alpha)}\frac{1}{\sqrt[q]{q\lambda}\|e_k\|_\lambda}\\ &\quad+\frac{\|L+S\|\|B\|\|\omega\|}{\lambda\Gamma(\alpha)}\sqrt[p]{\frac{1}{1-p(1-\alpha)}}\ t^{\frac{1}{p}-(1-\alpha)}\frac{1}{\sqrt[q]{q\lambda}\|e_k\|_\lambda},\end{aligned}$$

where $p\in\left(1,\frac{1}{1-\alpha}\right),\ \frac{1}{p}+\frac{1}{q}=1\quad(p,q>0)$.
This implies that

$$\begin{aligned}\|\Delta x_k\|_\lambda &\le \|L+S\|\|\psi\|\|e_k(0)\|+\frac{\|A\|}{\Gamma(\alpha)}\sqrt[p]{\frac{1}{1-p(1-\alpha)}}T^{\frac{1}{p}-(1-\alpha)}\frac{1}{\sqrt[q]{q\lambda}\|\Delta x_k\|_\lambda}\\ &\quad+\frac{\|L+S\|\|B\|\|\varphi\|}{\Gamma(\alpha)}\sqrt[p]{\frac{1}{1-p(1-\alpha)}}T^{\frac{1}{p}-(1-\alpha)}\frac{1}{\sqrt[q]{q\lambda}\|e_k\|_\lambda}\\ &\quad+\frac{\|L+S\|\|B\|\|\omega\|}{\lambda\Gamma(\alpha)}\sqrt[p]{\frac{1}{1-p(1-\alpha)}}T^{\frac{1}{p}-(1-\alpha)}\frac{1}{\sqrt[q]{q\lambda}\|e_k\|_\lambda}.\end{aligned}\tag{18}$$

Next,

$$\begin{aligned}e_{k+1}(t) &= e_k(t)-((I_N\otimes C)(x_{k+1}(t)-x_k(t))+(I_N\otimes D)(u_{k+1}(t)-u_k(t)))\\ &= e_k(t)-(I_N\otimes C)\ (x_{k+1}(t)-x_k(t))\\ &\quad-(I_N\otimes D)\left(((L+S)\otimes\varphi)e_k(t)+((L+S)\otimes\omega)\int_0^t e_k(\tau)d\tau\right)\\ &= (I_{mN}-(L+S)\otimes D\varphi)\ e_k(t)\\ &\quad-(I_N\otimes C)\ \ (x_{k+1}(t)-x_k(t))\ \ -((L+S)\otimes D\omega)\ \int_0^t e_k(\tau)d\tau.\end{aligned}$$

Thus,

$$\begin{aligned}\|e_{k+1}(t)\| &\le \|I_{mN}-(L+S)\otimes D\varphi\|\|e_k(t)\|+\|C\|\|\Delta x_k(t)\|\\ &\quad+\|L+S\|\|D\|\|\omega\|\int_0^t\|e_k(\tau)\|d\tau\\ &\le \|I_{mN}-(L+S)\otimes D\varphi\|\|e_k(t)\|+\|C\|\|\Delta x_k(t)\|\\ &\quad+\|L+S\|\|D\|\|\omega\|\frac{e^{\lambda t}-1}{\lambda}\|e_k\|_\lambda.\end{aligned}$$

Further, we have

$$\|e_{k+1}\|_\lambda \le \left(\|I_{mN} - (L+S) \otimes D\varphi\| + \|L+S\|\|D\|\|K\| \frac{1-e^{-\lambda T}}{\lambda} \right) \|e_k\|_\lambda + \|C\|\|\Delta x_k\|_\lambda. \quad (19)$$

Linking (18) and (19) for some sufficient large λ, we have

$$\|e_{k+1}\|_\lambda \le \|I_{mN} - (L+S) \otimes D\varphi\|\|e_k\|_\lambda + \|C\|\|L+S\|\|\psi\|\|e_k(0)\|. \quad (20)$$

Obviously, by Theorem 3.1 we have $\lim_{k\to\infty}\|e_k(0)\| \to 0$ via (9). Finally, from (10) and (20), we deduce that $\lim_{k\to\infty}\|e_k\|_\lambda = 0$. The proof is completed.

Remark 3.4. Obviously, one can repeat the above procedure in Theorems 3.1 and 3.3 to derive the same convergence results for the case of α=1.

4 | SIMULATION EXAMPLES

We consider a network of 5 agents to illustrate the efficacy of the proposed consensus scheme.

Figure 1 shows information flow among agents. Vertex 0 represents the virtual leader. It has directed edges to agents 1 and 3. The communication among followers is directed. We adopt 0−1 weighting, and the Laplacian for followers is

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

and S=$diag$(1,0,1,0).

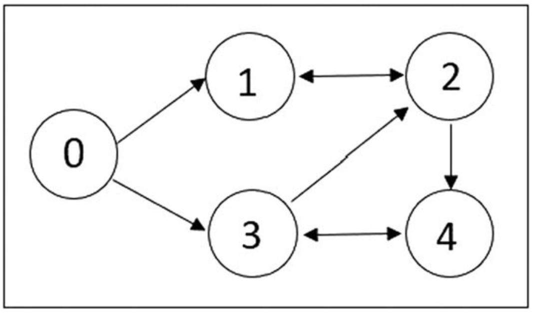

FIGURE 1 Directed communication topology among agents in the network

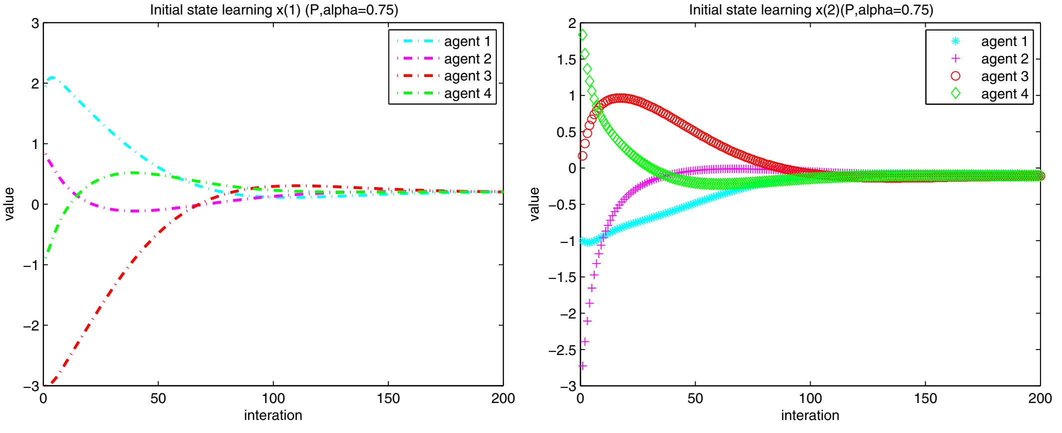

FIGURE 2 Initial state profile vs iteration number in Example 4.1 [Colour figure can be viewed at wileyonlinelibrary.com]

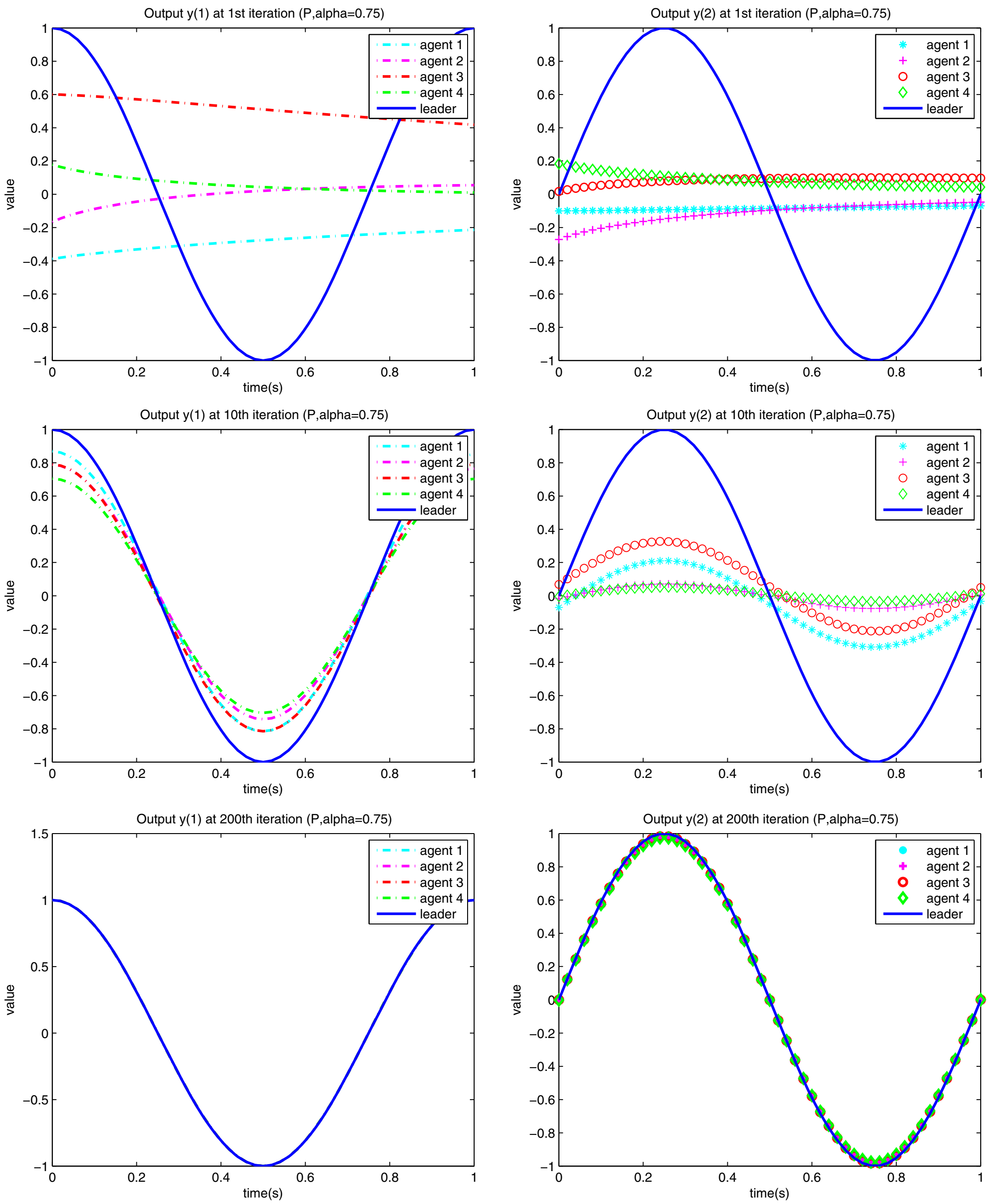

FIGURE 3 Output trajectory of all agents at different iterations in Example 4.1 [Colour figure can be viewed at wileyonlinelibrary.com]

In this section, we set α=0.75 and the norm of the tracking errors in each iteration is designated 2-norm in the following examples. The initial state at first iteration is chosen as $x_1=[2,-1]^T$, $x_2=[1,-3]^T$, $x_3=[-3,0]^T$, and $x_4=[-1,2]^T$. The desire initial state is unique $x_d=0$. The initial control signal $u_{1,j}=0, j=1,2,3,4$, for all agents.

Example 4.1. Consider the ith agent model of fractional order as follows:

$$\begin{cases} {}_0^cD_t^\alpha x_i(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x_i(t) + \begin{bmatrix} 0.1 & 0 \\ 0 & -0.2 \end{bmatrix} u_i(t), \\ y_i(t) = \begin{bmatrix} -0.2 & 0 \\ 0 & 0.1 \end{bmatrix} x_i(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0.2 \end{bmatrix} u_i(t), \end{cases} \tag{21}$$

and the desired reference trajectory $y_d = \begin{bmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{bmatrix}, \quad \forall t \in [0, 1]$. To verify the contraction conditions in Theorem 3.1, we select the learning gain matrix

$$\varphi = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad \psi = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

Clearly, $\|I-(L+S)\otimes D\varphi\|=0.9822<1$ and $\|I-(L+S)\otimes C\psi-(L+S)\otimes D\varphi\|=0.9734<1$.
Thus, the result in Theorem 3.1 is valid for Example 4.1.

Figure 2 shows the agents' initial state learning. In Example 4.1, the desired initial state converges to the desired initial state asymptomatically around 150th iteration. Figure 3 shows the agents' output at the 1st, 10th, and 200th iterations. With the iteration grows, all agents' output converges to the desired trajectory. Figure 4 depicts the agents' tracking errors in each iteration, which can be approximated to desired trajectory in a finite time interval.

Example 4.2. We still consider (21) with the identical desired reference trajectory.

To verify the contraction conditions in Theorem 3.3, we keep to use φ and ψ defined in Example 4.1 and elect the learning gain matrix $\omega = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}$. Clearly, $\|I-(L+S)\otimes D\varphi\|=0.9822<1$ and $\|I-(L+S)\otimes C\psi-(L+S)\otimes D\varphi\|=0.9734<1$.Thus, the result in Theorem 3.3 is valid for Example 4.2.

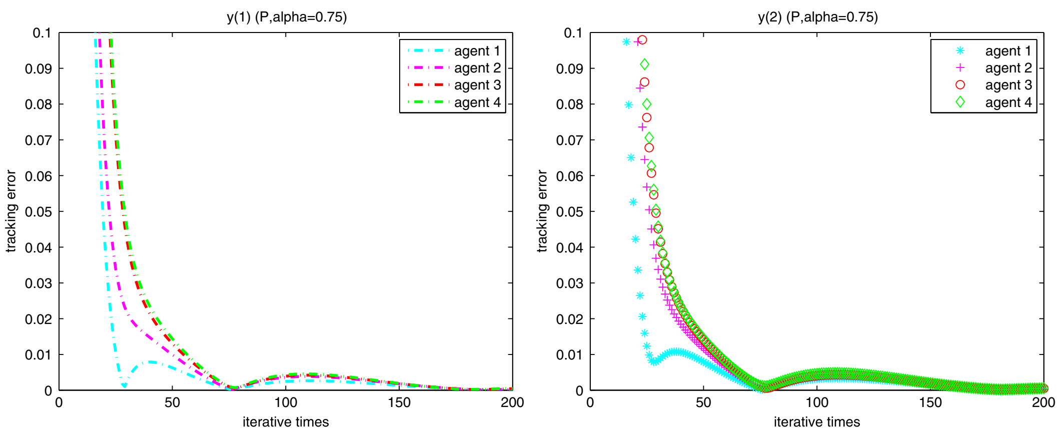

FIGURE 4 The 2-norm of the tracking errors for all agents in each iteration in Example 4.1 [Colour figure can be viewed at wileyonlinelibrary.com]

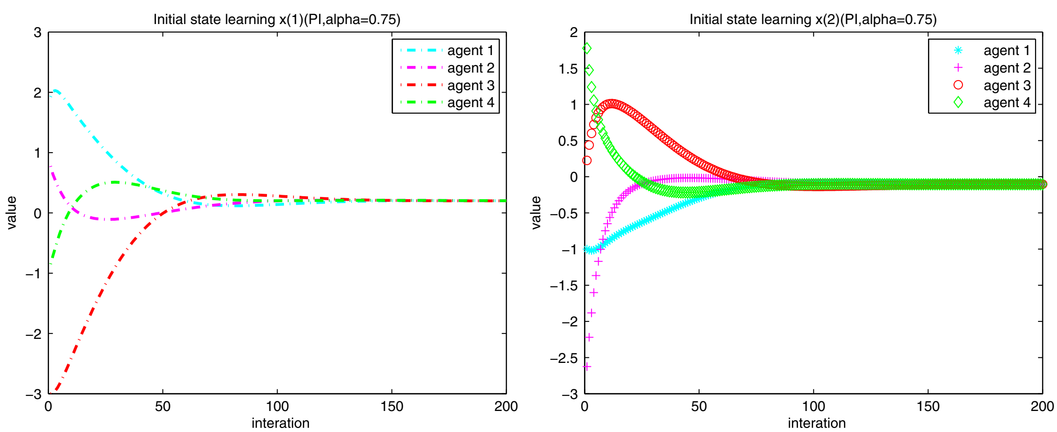

FIGURE 5 Initial state profile vs iteration number in Example 4.2 [Colour figure can be viewed at wileyonlinelibrary.com]

Figure 5 shows the agents' initial state learning. In Example 4.2, the desired initial state converges to the desired initial state asymptomatically around 150th iteration. Figure 6 shows the agents' output at the 1st, 10th, and 200th iterations. With the iteration grows, all agents' output converges to the desired trajectory. Figure 7 depicts the agents' tracking errors in each iteration, which can be approximated to desired trajectory in a finite time interval.

Remark 4.3. Table 1 shows the effectiveness of the proposed algorithms for both fractional-order and integral-order systems (α=0.75 and α=1). We observe that the tracking errors of fractional-order systems are much smaller than those of integral-order systems due to order difference between the system and control law. The error value of fractional-order systems in the 200th iteration is always smaller than 0.001, while that of integral-order systems is sometimes smaller than 0.001. Moreover, for the same fractional order (α=0.75 in this example), Table 1 also displays that *PI*-type learning law surpasses *P*-type learning law.

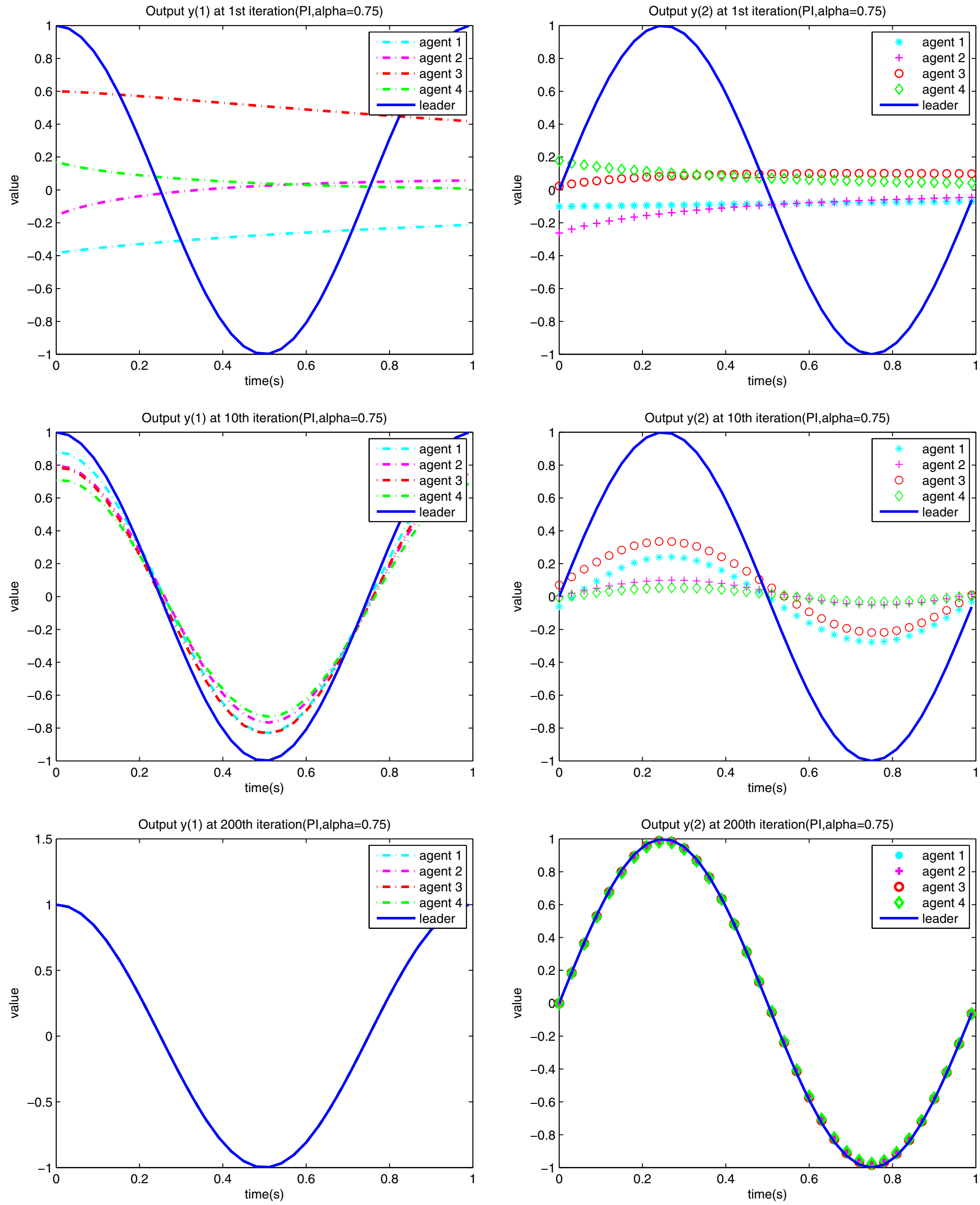

FIGURE 6 Output trajectory of all agents at different iterations in Example 4.2 [Colour figure can be viewed at wileyonlinelibrary.com]

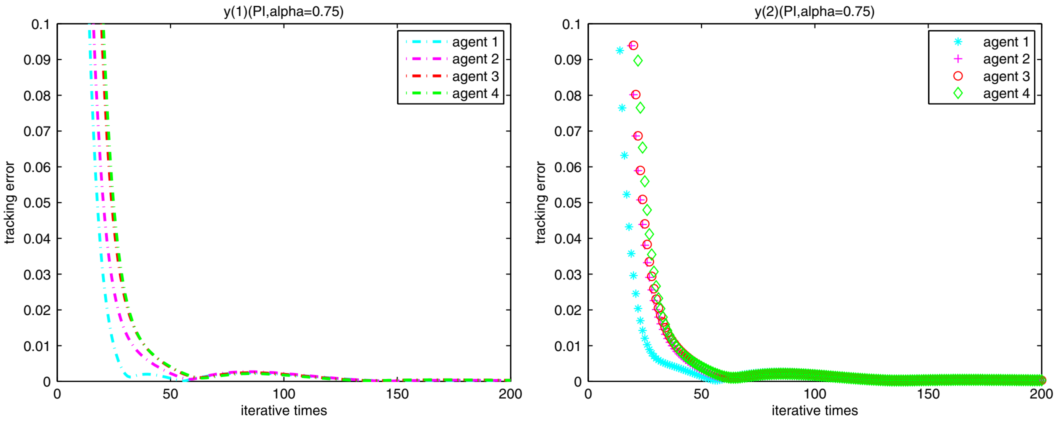

FIGURE 7 The 2-norm of the tracking errors for all agents in each iteration in Example 4.2 [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 1 Comparison of tracking error of different learning laws in 200th iteration

Agent	*P*-Type Learning Law		*PI*-Type Learning Law	
	α=0.75	α=1	α=0.75	α=1
1	0.000104	0.000403	0.000182	0.001412
	0.000167	0.000648	0.000220	0.001659
2	0.000255	0.000920	0.000254	0.001796
	0.000336	0.001131	0.000283	0.001863
3	0.000411	0.001322	0.000308	0.001904
	0.000476	0.001526	0.000310	0.001882
4	0.000511	0.001686	0.000304	0.001842
	0.000531	0.001837	0.000294	0.001790

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ORCID

Dahui Luo http://orcid.org/0000-0001-6720-2289
JinRong Wang http://orcid.org/0000-0002-6642-1946
Dong Shen http://orcid.org/0000-0003-1063-1351

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