Iterative learning control for fractional-order multi-agent systems

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Received 28 January 2018; received in revised form 26 February 2019; accepted 2 June 2019
Available online 8 June 2019

Abstract

In this paper, we apply iterative learning control to both linear and nonlinear fractional-order multi-agent systems to solve consensus tacking problem. Both fixed and iteration-varying communicating graphs are addressed in this paper. For linear systems, a $PD^\alpha$-type update law with initial state learning mechanism is introduced by virtue of the memory property of fractional-order derivative. For nonlinear systems, a $D^\alpha$-type update law with forgetting factor and initial state learning is designed. Sufficient conditions for both linear and nonlinear systems are established to guarantee all agents achieving the asymptotic output consensus. Simulation examples are provided to verify the proposed schemes.

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\* This work was supported by the National Natural Science Foundation of China (grant numbers 11661016; 61673045), Science and Technology Program of Guizhou Province ([2017]5788-10), Major Research Project of Innovative Group in Guizhou Education Department ([2018]012), Slovak Research and Development Agency under the contract No. APVV-14-0378 and by the Slovak Grant Agency VEGA No. 2/0153/16 and No. 1/0078/17.

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https://doi.org/10.1016/j.jfranklin.2019.06.001
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1. Introduction

As is well known, Arimoto et al. offered the concept of iterative learning control (ILC) in 1980s, which has been wildly applied to robotics, process control, and biological systems to obtain a perfect tracking performance in a finite time interval. Based on the basic cognition that learning can help to improve the control performance as the iteration number increases, ILC has been deeply investigated from both theoretical analysis and practical applications in the past three decades. On the other hand, fractional calculus has a history over three hundreds year, which is a natural extension of the classical calculus. We note that fractional-order systems have numerous applications in viscoelasticity, electrochemistry and control. Indeed, there are some previous studies reported on ILC for fractional-order systems [1–6]. However, the research on this issue is still at the initial stage and many open problems exists.

Multi-agent systems (MAS) have been found considerable applications in cross-disciplinary nature [7–9]. Different from the integer order MASs, the Lyapunov stability theory is invalid in fractional-order MAS (FOMAS) due the memory property of fractional order derivatives. It is difficult to adopt the standard framework to derive extended results. Note that there does exist time delay phenomena when the agents work in some extreme environments, leading to the system instability. FOMAS provides a favorite model to describe MASs with memory property. Therefore, it is of great significance to study consensus problem of FOMAS. As a natural extension, FOMAS have attracted much attention on the coordination and control problems, and some primary results can be found in [10–12]. For more recent contributions on containment control, adaptive robust tracking control and optimal control, one can also refer to [13–21]. However, no result on the learning consensus of FOMAS has been reported in the existing literature. This observation motivates us to consider possible strategies for improving consensus performance of all agents. Indeed, learning in MAS has been one of the most fertile grounds for interaction between game theory and artificial intelligence [22]. Therefore, it is of great significance to investigate the learning coordination problem of FOMAS where certain memory property is involved.

Noting that ILC is a well established learning control strategy, in which the past tracking information is utilized to improve the tracking performance of the current iteration, it is a sound idea to apply ILC technology to the learning consensus tracking problem of FOMAS. We should note that, although none work on ILC for FOMAS has been reported, the ILC has been successfully applied to the coordination problem of conventional MAS (see [23–36]), where the communication graph can be fixed and directed. Comparing with the conventional MAS case, we note that FOMAS will present certain interleaving effect among agents. Such effect would make the tracking performance be involved with memory property that should be carefully studied.

In this paper, we apply $PD^\alpha$-type ILC update law to achieve learning consensus for linear FOMAS. Incorporated with an initial state learning mechanism, we present sufficient conditions for asymptotic output consensus of all agents as the iteration number increases. It is emphasized that the memory property of fractional-order derivative is carefully studied in this paper for showing the learning ability of each agent. We first derive a convergence theorem for a fixed communication topology and then extend the results to iteration-varying case. Both cases demonstrate that the proposed scheme behaves effective in achieving consensus along the iteration axis. Further, we apply $D^\alpha$-type ILC update law with a forgetting factor to achieve learning consensus for nonlinear FOMAS. To the best knowledge of authors, this paper presents the first result on learning consensus of FOMAS.
The rest of this paper is organized as follows. In Section 2, we present the problem formulation. The main results for linear and nonlinear cases are provided in Section 3, where both fixed and iteration-varying communication graphs are discussed. Simulation examples are given in Section 4 to illustrate the effectiveness of the proposed scheme. Section 5 concludes this paper.

2. Notations and problem formulation

Denote both vector Euclidian norm and its compatible matrix norm by \( \| \cdot \| \). The standard \( \lambda - \)norm for a function \( g: [0, T] \rightarrow \mathbb{R}^n \) is defined as \( \| g \|_\lambda = \sup_{t \in [0, T]} e^{-\lambda t} \| g(t) \| \) where \( \lambda > 0 \). We collect some knowledge of graph theory to formulate MAS (see [24,29]). Let \( G = (V, E, \tilde{A}) \) be a weighted directed graph, \( V = \{ 1, 2, \ldots, N \} \) the set of vertices, \( E \subseteq V \times V \) the set of edges and \( \tilde{A} \) the adjacency matrix. We write a pair \((i,j) \in E\) as a direct edge from \( i \) to \( j \); that is, agent \( j \) can receive information from agent \( i \). Then, we say \( i \) is the parent of \( j \), and \( j \) is the child of \( i \).

The set of neighbors of \( i \)th agent is denoted by \( \mathcal{N}_i = \{ j \in V : (j, i) \in E \} \). \( \tilde{A} = (\tilde{a}_{i,j}) \in \mathbb{R}^{N \times N} \) is the weighted adjacency matrix of \( G \) with \( \tilde{a}_{i,j} \geq 0 \). In particular, \( \tilde{a}_{i,j} = 1 \) if \((i,j) \in E\) and \( i \neq j \) and 0 otherwise. Denote \( d_{i}^{in} = \sum_{j=1}^{N} \tilde{a}_{i,j} \) the in-degree of vertex \( i \), \( D = diag(d_1^{in}, \ldots, d_N^{in}) \) and \( L = D - \tilde{A} \) be the Laplacian of \( G \). Denote \( \otimes \) by Kronecker product. For some matrices \( A, B, C \) and \( D \) of appropriate dimensions, \( k(A \otimes B) = A \otimes kB, (A + B) \otimes C = A \otimes C + B \otimes C, (A \otimes B)(C \otimes D) = AC \otimes BD \), and \( \| A \otimes B \| = \| A \| \cdot \| B \| \).

Consider a network of \( N \) agents, of which the communication topology is described by \( G = (V, E, \tilde{A}) \) with \( \tilde{A} = (\tilde{a}_{i,j}) \). The \( i \)th agent is governed by the following linear fractional-order model:

\[
\mathbf{D}^{\alpha}_t x_i(t) = A x_i(t) + B u_i(t), \quad x_i(0) = x_{i,0}, \quad i \in V.
\]

\[
y_i(t) = C x_i(t),
\]

where \( \mathbf{D}^{\alpha}_t x_i(t) := \frac{1}{\Gamma(1-\alpha)} \int_{0}^{t} \frac{x_i(s) - x_i(0)}{(t-s)^\alpha} ds \) denotes the Caputo fractional derivative of order \( \alpha \) for \( x_i \) with the lower limit zero (see [37]), \( x_i \in \mathbb{R}^{n \times 1} \) is the state vector, \( y_i \in \mathbb{R}^{m \times 1} \) is the output vector, \( u_i \in \mathbb{R}^{m \times 1} \) is the control input, and \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{m \times n} \) are the constant matrices. Throughout this paper, we assume the input/output coupling matrix is of full-column rank, i.e., \( \text{rank}(CB) = m \).

Moreover, we also consider \( N \) dynamic agents, where the \( i \)th agents at \( k \)th iteration is governed by the following nonlinear time-invariant model:

\[
\mathbf{D}^{\alpha}_t x_{k,i}(t) = f(x_{k,i}(t), t) + B u_{k,i}(t), \quad i = 1, 2, \ldots, N; t \in [0, T]
\]

\[
y_{k,i}(t) = C x_{k,i}(t).
\]

Here, the nonlinear continues function \( f_{k,i} \) is required to satisfy the globally Lipschitz continuous condition specified later.

Let \( y_{d}(t), t \in [0, T] \) be the desired sufficiently smooth trajectory for consensus tracking. This desired trajectory can be accessible only by a subset of followers. In the communication topology, we treat \( y_{d}(t) \) as the leader and index it by vertex 0. Then the extended topology including both leader and all followers can be expressed as \( G^{+} = (V \cup \{ 0 \}, E^{+}, A^{+}) \), in which \( 0 \) is the leader, \( E^{+} \) is the edge set and \( A^{+} \) is the weighted adjacency matrix of \( G^{+} \).

The control objective of this paper is to design distributed learning strategy for the FOMAS such that all agents can achieve asymptotical consensus during the specified finite time interval.
as the iteration number increases. This control objective is accomplished in the next section by incorporating with an initial state learning mechanism.

3. Main results

The section contains two subsections. In Section 3.1, the distributed PD-type updating rule and its convergence properties are adequately revealed for linear FOMASs with fixed communicating graph and iteration-varying communicating graph, respectively. In Section 3.2, the D-type updating rule with a forgetting factor is emphatically discussed for nonlinear FOMASs with fixed communicating graph.

3.1. Convergence analysis for linear FOMASs

3.1.1. Convergence analysis with fixed communicating graph

Let \( \eta_k(t) \) be the available information at the \((k+1)\)th iteration for the \(j\)th agent, where \(k\) denotes the iteration number and \(j\) denotes the agent index. Consider

\[
\eta_{i,j}(t) = \sum_{\omega \in N_j} \tilde{a}_{j,\omega}(y_{i,\omega}(t) - y_{i,j}(t)) + s_j(y_d(t) - y_{i,j}(t)),
\]

where \(s_j\) is equal to 1 if the \(j\)th agent can access the desired trajectory and 0 otherwise.

For Eq. (1) with Eq. (2), we consider the PD-type ILC updating law:

\[
u_{k+1,j}(t) = u_{k,j}(t) + W_1 \odot D_{i,j}^\alpha \eta_{k,j}(t) + W_2 \eta_{k,j}(t),
\]

with the initial state learning mechanism:

\[
x_{k+1,j}(0) = x_{k,j}(0) + BW_1 \eta_{k,j}(0),
\]

where \(\eta_{k,j}\) is defined in Eq. (4), and \(W_1 \in \mathbb{R}^{m \times m}\) and \(W_2 \in \mathbb{R}^{m \times m}\) denote learning gain matrices.

Let \(e_{k,j}(t) = y_d(t) - y_{k,j}(t)\) be the tracking error. Then we write Eq. (4) as

\[
\eta_{i,j}(t) = \sum_{\omega \in N_j} \tilde{a}_{j,\omega}(e_{i,j}(t) - e_{i,\omega}(t)) + s_j e_{i,j}(t)
\]

in term of errors.

For the \(k\)th iteration, we define the column stack vectors:

\[
\eta_k(t) = [\eta_{k,1}(t)^T, \eta_{k,2}(t)^T, \ldots, \eta_{k,N}(t)^T]^T,
\]

\[
x_k(t) = [x_{k,1}(t)^T, x_{k,2}(t)^T, \ldots, x_{k,N}(t)^T]^T,
\]

\[
u_k(t) = [u_{k,1}(t)^T, u_{k,2}(t)^T, \ldots, u_{k,N}(t)^T]^T,
\]

\[
e_k(t) = [e_{k,1}(t)^T, e_{k,2}(t)^T, \ldots, e_{k,N}(t)^T]^T.
\]

Thus, linking Eqs. (5), (6) and (7) via Kronecker product, one obtains

\[
\eta_k = ((L + S) \otimes I_n) e_k(t),
\]

\[
u_{k+1}(t) = u_k(t) + ((L + S) \otimes W_1) \odot D_{i,j}^\alpha e_k(t) + ((L + S) \otimes W_2) e_k(t),
\]

and

\[
x_{k+1}(0) = x_k(0) + ((L + S) \otimes BW_1) e_k(0),
\]
where \( L \) denotes the graph Laplacian of \( G \), and \( S = \text{diag}(s_1, s_2, \ldots, s_N) \), \( s_i \geq 0 \), \( i = 1, 2, \ldots, N \) is associated with \( G^+ \).

We are now ready to present the first main theorem of this paper.

**Theorem 3.1.** For Eq. (1) with Eq. (2), considering Eqs. (5) and (6), the consensus tracking error \( e_i(t) \to 0 \) as iteration number \( i \to \infty \), i.e. \( \lim_{i \to \infty} y_{i,j}(t) = y_d(t) \) for all \( t \in [0, T] \), provided that

\[
\|I_{nN} - (L + S) \otimes CBW_1\| < 1. \tag{11}
\]

**Proof.** Note that the tracking error of the \( j \)th agent is

\[
e_{k+1,j}(t) = y_d(t) - y_{k+1,j}(t)
\]

\[
= y_d(t) - y_{k,j}(t) - (y_{k+1,j}(t) - y_{k,j}(t))
\]

\[
= e_{k,j}(t) - (y_{k+1,j}(t) - y_{k,j}(t)). \tag{12}
\]

Based on Eq. (12), we can write the tracking errors for all agents as follows

\[
e_{k+1}(t) = e_k(t) - (I_N \otimes C)(x_{k+1}(t) - x_k(t)). \tag{13}
\]

Firstly, we need to derive the agents’ state trajectory via Eqs. (9) and (10) on \([0, T]\). In fact,

\[
x_{k+1}(t) = x_{k+1}(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (I_N \otimes A)x_{k+1}(\tau) + (I_N \otimes B)u_{k+1}(\tau) d\tau
\]

\[
= x_k(0) + \left( (L + S) \otimes BW_1 \right) e_k(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (I_N \otimes A)x_k(\tau) + (I_N \otimes B)u_k(\tau) d\tau
\]

\[
+ \frac{I_N \otimes A}{\Gamma(\alpha)} \int_0^t x_{k+1}(\tau) - x_k(\tau) d\tau + \frac{I_N \otimes B}{\Gamma(\alpha)} \int_0^t u_{k+1}(\tau) - u_k(\tau) d\tau
\]

\[
= x_k(t) + \left( (L + S) \otimes BW_1 \right) e_k(0) + \frac{I_N \otimes A}{\Gamma(\alpha)} \int_0^t x_{k+1}(\tau) - x_k(\tau) d\tau
\]

\[
+ \frac{(L + S) \otimes BW_1}{\Gamma(\alpha)} \int_0^t \frac{D^\alpha_{0} e_k(\tau)}{\Gamma(\alpha)} d\tau + \frac{(L + S) \otimes BW_2}{\Gamma(\alpha)} \int_0^t \frac{e_k(\tau)}{\Gamma(\alpha)} d\tau
\]

\[
= x_k(t) + \left( (L + S) \otimes BW_1 \right) e_k(0) + \frac{I_N \otimes A}{\Gamma(\alpha)} \int_0^t x_{k+1}(\tau) - x_k(\tau) d\tau
\]

\[
+ \frac{(L + S) \otimes BW_1}{\Gamma(\alpha)} \cdot (e_k(t) - e_k(0)) + \frac{(L + S) \otimes BW_2}{\Gamma(\alpha)} \int_0^t \frac{e_k(\tau)}{\Gamma(\alpha)} d\tau
\]

\[
= x_k(t) + \left( (L + S) \otimes BW_1 \right) e_k(0) + \frac{I_N \otimes A}{\Gamma(\alpha)} \int_0^t x_{k+1}(\tau) - x_k(\tau) d\tau
\]

\[
+ \frac{(L + S) \otimes BW_2}{\Gamma(\alpha)} \int_0^t \frac{e_k(\tau)}{\Gamma(\alpha)} d\tau,
\]

where we use the fact \( R^L_{0} I_{t}^{\alpha} (D_{0}^{\alpha} f)(t) = f(t) - f(0) \) (see [37]), the symbol \( R^L_{0} I_{t}^{\alpha} f := \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{\Gamma(\alpha)} d\tau \) denotes Riemann–Liouville integral of a function \( f \) of fractional order.
Taking \( \Delta x_k(t) = x_{k+1}(t) - x_k(t) \)

\[
\begin{align*}
\Delta x_k(t) &= (L + S) \otimes BW_1 e_k(t) + \frac{(I_N \otimes A)}{\Gamma(\alpha)} \int_0^t \frac{\Delta x_k(\tau)}{(t - \tau)^{1-\alpha}} d\tau \\
&\quad + \frac{(L + S) \otimes BW_2}{\Gamma(\alpha)} \int_0^t \frac{e_k(\tau)}{(t - \tau)^{1-\alpha}} d\tau.
\end{align*}
\]

(15)

Taking norm on both side of Eq. (15), we have

\[
\| \Delta x_k(t) \| \leq \| L + S \| \| B \| \| W_1 \| \| e_k(t) \| + \frac{\| I_N \otimes A \|}{\Gamma(\alpha)} \int_0^t \frac{\| \Delta x_k(\tau) \|}{(t - \tau)^{1-\alpha}} d\tau \\
&\quad + \frac{\| L + S \| \| B \| \| W_2 \|}{\Gamma(\alpha)} \int_0^t \frac{\| e_k(\tau) \|}{(t - \tau)^{1-\alpha}} d\tau.
\]

(16)

Further, multiplying the factor \( e^{-\lambda t} \) on both side of Eq. (16), one has

\[
\| \Delta x_k(t) \| e^{-\lambda t} \leq \| L + S \| \| B \| \| W_1 \| \| e_k(t) \| e^{-\lambda t} + \frac{\| I_N \otimes A \|}{\Gamma(\alpha)} \int_0^t \frac{e^{\lambda \tau} e^{-\lambda \tau} \| \Delta x_k(\tau) \|}{(t - \tau)^{1-\alpha}} d\tau \\
&\quad + \frac{\| L + S \| \| B \| \| W_2 \|}{\Gamma(\alpha)} \int_0^t \frac{e^{\lambda \tau} e^{-\lambda \tau} \| e_k(\tau) \|}{(t - \tau)^{1-\alpha}} d\tau \\
&\leq \| L + S \| \| B \| \| W_1 \| \| e_k(t) \| e^{-\lambda t} + \frac{\| I_N \otimes A \|}{\Gamma(\alpha)} \int_0^t \frac{e^{\lambda \tau}}{(t - \tau)^{1-\alpha}} d\tau \| \Delta x_k \|_\lambda \\
&\quad + \frac{\| L + S \| \| B \| \| W_2 \|}{\Gamma(\alpha)} e^{-\lambda t} \int_0^t \frac{e^{\lambda \tau}}{(t - \tau)^{1-\alpha}} d\tau \| e_k \|_\lambda.
\]

Note that for some \( p \in (1, \frac{1}{1-\alpha}) \),

\[
\int_0^t (t - \tau)^{\alpha-1} e^{\lambda \tau} d\tau \leq \sqrt{\frac{1}{1 - p(1 - \alpha)}} \frac{1}{\sqrt{q \lambda}^\alpha} e^{\lambda t},
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( p, q > 0 \). This gives

\[
e^{-\lambda t} \int_0^t (t - \tau)^{\alpha-1} e^{\lambda \tau} d\tau \leq \sqrt{\frac{1}{1 - p(1 - \alpha)}} \frac{1}{\sqrt{q \lambda}^\alpha} e^{\lambda t}.
\]

Finally, we obtain

\[
\| \Delta x_k(t) \| e^{-\lambda t} \leq \| L + S \| \| B \| \| W_1 \| \| e_k(t) \| e^{-\lambda t} + \frac{\| I_N \otimes A \|}{\Gamma(\alpha)} \sqrt{\frac{1}{1 - p(1 - \alpha)}} \frac{1}{\sqrt{q \lambda}^\alpha} \| \Delta x_k \|_\lambda \\
&\quad + \frac{\| L + S \| \| B \| \| W_2 \|}{\Gamma(\alpha)} \sqrt{\frac{1}{1 - p(1 - \alpha)}} \frac{1}{\sqrt{q \lambda}^\alpha} \| e_k \|_\lambda.
\]

(17)

Taking supremum for Eq. (17) on \([0, T]\), we have

\[
\| \Delta x_k \|_\lambda \leq \| L + S \| \| B \| \| W_1 \| \| e_k \|_\lambda + \frac{\| I_N \otimes A \|}{\Gamma(\alpha)} \sqrt{\frac{1}{1 - p(1 - \alpha)}} T^{\frac{1}{\alpha} - (1-\alpha)} \frac{1}{\sqrt{q \lambda}^\alpha} \| \Delta x_k \|_\lambda
\]
\[
\frac{\|L + S\|\|B\|\|W_2\|}{\Gamma(\alpha)} \sqrt{\frac{1}{1 - p(1 - \alpha)}} T^{\frac{1}{\alpha} - (1 - \alpha)} \frac{1}{\sqrt{qT}} \|e_k\|_\lambda.
\]

Obviously, for some \( \lambda \) large enough, i.e.,
\[
\lambda \geq \Theta_1 := \frac{1}{q} \left( \frac{1}{\Gamma(\alpha)} \right) \left( 2\|I_N \otimes A\| + \frac{\|W_2\|}{\|W_1\|} \right) \sqrt{\frac{1}{1 - p(1 - \alpha)}} T^{\frac{1}{\alpha} - (1 - \alpha)} q,
\]
we have
\[
\|\Delta x_k\|_\lambda \leq 2\|L + S\|\|B\|\|W_1\|\|e_k\|_\lambda.
\]

Substituting Eq. (15) into Eq. (13),
\[
e_{k+1}(t) = e_k(t) - (I_N \otimes C) \left( (L + S) \otimes BW_1 \right) e_k(t) + \frac{(I_N \otimes A)}{\Gamma(\alpha)} \int_0^t \frac{\Delta x_k(\tau)}{(t - \tau)^{1 - \alpha}} d\tau
\]
\[
+ \frac{(L + S) \otimes BW_2}{\Gamma(\alpha)} \int_0^t \frac{e_k(\tau)}{(t - \tau)^{1 - \alpha}} d\tau,
\]
and taking norm of Eq. (21) to derive
\[
\|e_{k+1}(t)\| \leq \|I_{mN} - (L + S) \otimes CBW_1\|e_k(t) + \frac{\|I_N \otimes CA\|}{\Gamma(\alpha)} \int_0^t \frac{\Delta x_k(\tau)}{(t - \tau)^{1 - \alpha}} d\tau
\]
\[
+ \frac{\|L + S\|\|C\|\|B\|\|W_2\|}{\Gamma(\alpha)} \int_0^t \frac{e_k(\tau)}{(t - \tau)^{1 - \alpha}} d\tau.
\]

Then, multiplying by the factor \( e^{-\lambda t} \) on both side of Eq. (23),
\[
\|e_{k+1}(t)\| e^{-\lambda t} \leq \|I_{mN} - (L + S) \otimes CBW_1\|\|e_k(t)\| e^{-\lambda t}
\]
\[
+ \frac{\|I_N \otimes CA\|}{\Gamma(\alpha)} e^{-\lambda t} \int_0^t (t - \tau)^{\alpha - 1} e^{\lambda \tau} e^{-\lambda t} \|\Delta x_k(\tau)\| d\tau
\]
\[
+ \frac{\|L + S\|\|C\|\|B\|\|W_2\|}{\Gamma(\alpha)} e^{-\lambda t} \int_0^t (t - \tau)^{\alpha - 1} e^{\lambda \tau} e^{-\lambda t} \|e_k(\tau)\| d\tau
\]
\[
\leq \|I_{mN} - (L + S) \otimes CBW_1\|\|e_k(t)\| e^{-\lambda t}
\]
\[
+ \frac{\|I_N \otimes CA\|}{\Gamma(\alpha)} e^{-\lambda t} \int_0^t (t - \tau)^{\alpha - 1} e^{\lambda \tau} d\tau \|\Delta x_k\|_\lambda
\]
\[
+ \frac{\|L + S\|\|C\|\|B\|\|W_2\|}{\Gamma(\alpha)} e^{-\lambda t} \int_0^t (t - \tau)^{\alpha - 1} e^{\lambda \tau} d\tau \|e_k\|_\lambda
\]
\[
\leq \|I_{mN} - (L + S) \otimes CBW_1\|\|e_k(t)\| e^{-\lambda t}
\]
\[ + \frac{\|I_N \otimes CA\|}{\Gamma(\alpha)} \sqrt{\frac{1}{1 - p(1 - \alpha)} T_\alpha^{1-(1-\alpha)} \frac{1}{\sqrt{q\lambda}}} \|\Delta x_k\|_\lambda \]

\[ + \frac{\|L + S\|\|C\|\|B\|\|W_2\|}{\Gamma(\alpha)} \sqrt{\frac{1}{1 - p(1 - \alpha)} T_\alpha^{1-(1-\alpha)} \frac{1}{\sqrt{q\lambda}}} \|e_k\|_\lambda, \]

which implies that

\[ \|e_{k+1}\|_\lambda \leq \left( \|I_{mN} - (L + S) \otimes CBW_1\| + \frac{\|L + S\|\|C\|\|B\|\|W_2\|}{\Gamma(\alpha)} \sqrt{\frac{1}{1 - p(1 - \alpha)} T_\alpha^{1-(1-\alpha)} \frac{1}{\sqrt{q\lambda}}} \|e_k\|_\lambda \right)^q. \]

From Eqs. (20) and (25) we conclude that

\[ \|e_{k+1}\|_\lambda \leq \left( \|I_{mN} - (L + S) \otimes CBW_1\| + \frac{2\|I_N \otimes CA\|\|W_1\| + \|W_2\|}{\Gamma(\alpha)} \sqrt{\frac{1}{1 - p(1 - \alpha)} T_\alpha^{1-(1-\alpha)} \frac{1}{\sqrt{q\lambda}}} \|e_k\|_\lambda \right)^q. \]

Finally, for some sufficient large \( \lambda \), i.e., satisfying Eq. (19) and the following one

\[ \lambda \geq \Theta_2 := \frac{1}{q} \left( \frac{2\|L + S\|\|B\|}{\Gamma(\alpha)} \sqrt{\frac{1}{1 - p(1 - \alpha)} T_\alpha^{1-(1-\alpha)} \frac{2\|I_N \otimes CA\|\|W_1\| + \|W_2\|}{1 - \|I_{mN} - (L + S) \otimes CBW_1\|}} \right)^q, \]

via Eqs. (11) and (26), we deduce

\[ \|e_{k+1}\|_\lambda \leq \frac{1 + \|I_{mN} - (L + S) \otimes CBW_1\|}{2} \|e_k\|_\lambda, \]

implying

\[ \|e_k\|_\lambda \leq \left( \frac{1 + \|I_{mN} - (L + S) \otimes CBW_1\|}{2} \right)^{k-1} \|e_1\|_\lambda, \forall k \geq 1, \]

so \( \lim_{k \to \infty} \|e_k\|_\lambda = 0. \) □

**Remark 3.2.** Since

\[ \|e_k(t)\| \leq \left( \frac{1 + \|I_{mN} - (L + S) \otimes CBW_1\|}{2} \right)^{k-1} \|e_1\|_\lambda e^{\lambda t}, \forall k \geq 1, \forall t \in [0, T] \]

for \( \lambda \) satisfying both Eqs. (19) and (27), the uniform norm convergence of \( e_k(t) \Rightarrow 0 \) on \([0, T]\) can be slow at the beginning of the iteration process. As a matter of fact, we have

\[ \|e_k(t)\| \leq \left( \frac{1 + \|I_{mN} - (L + S) \otimes CBW_1\|}{2} \right)^{k-k_0}, \forall t \in [0, T] \]
for any $k \geq k_0$ with
\[ k_0 = \max \left\{ 1, \left[ 1 - \frac{\ln \|e_1\| + \lambda T}{\ln \left( 1 + \frac{\|e_n\| - (L+S)\otimes CBW_1\|}{2} \right)} \right] \right\}, \]

where $\lceil \cdot \rceil$ is the floor function. We can take $\lambda = \max\{\Theta_1, \Theta_2\}$. We see that after $k_0$ iterations, the uniform norm convergence of $e_k(t) \to 0$ on $[0, T]$ is stabilizing.

**Remark 3.3.** Theorem 3.1 reveals that the $PD^\alpha$-type ILC update law is an effective approach to achieve the consensus performance for linear FOMAS (1) and (2). The sufficient condition for achieving the consensus performance. We note that the term $D^\alpha$ in Eq. (5) is necessary for linear FOMAS (1) and (2) according the causality principle. If the output Eq. (2) contains a direct control term, then one can refer to [17, Theorem 3.1] for related discussions.

### 3.1.2. Convergence analysis with iteration-varying communicating graph

In this section, we extend the fixed communication graph to an iteration-varying communication graph. Now we write Eqs. (8), (9) and (10) respectively in an iteration-varying form as follows:

\[ \eta_k = (H(k) \otimes I_m)e_k(t), \quad H(k) = L(k) + S(k), \]

\[ u_{k+1}(t) = u_k(t) + (H(k) \otimes W_1) D^\alpha_0 e_k(t) + (H(k) \otimes W_2)e_k(t), \tag{28} \]

\[ x_{k+1}(0) = x_k(0) + (H(k) \otimes BW_1)e_k(0), \tag{29} \]

where $L(k) \in \mathbb{R}^N$ is the Laplacian matrix in $k$th iteration, $S(k) = \text{diag}(s_1(k), s_2(k), \ldots, s_N(k))$, and $s_j(k)$ is equal to 1 if the $j$th agent can access the desired trajectory in $k$th iteration and 0 otherwise.

We now present the second main theorem of this paper.

**Theorem 3.4.** Suppose $\sup_k \|L(k)\| < \infty$ and $\sup_k \|S(k)\| < \infty$. For Eq. (1) with Eqs. (2), (28) and (29) are considered in all agents, the consensus tracking error $e_k(t) \to 0$ as iteration $k \to \infty$, i.e. $\lim_{k \to \infty} y_{k,j}(t) = y_d(t)$ for all $t \in [0, T]$, provided that

\[ \limsup_{k \to \infty} \|I_{mN} - H(k) \otimes CBW_1\| < 1. \tag{30} \]

**Proof.** The proof is similar to Theorem 3.1. Here we only point the major differences. Firstly, one can obtain the following agents’ state trajectory via Eqs. (28) and (29) on $[0, T]$:

\[ \triangle x_k(t) = \left( H(k) \otimes BW_1 \right) e_k(t) + \frac{(I_N \otimes A)}{\Gamma(\alpha)} \int_0^t \frac{\triangle x_k(\tau)}{(t - \tau)^{1-\alpha}} d\tau + \frac{H(k) \otimes BW_2}{\Gamma(\alpha)} \int_0^t \frac{e_k(\tau)}{(t - \tau)^{1-\alpha}} d\tau. \tag{31} \]

For Eq. (31), taking the $\lambda$-norm as well as estimate via Hölder inequality, one obtains

\[ \|\triangle x_k\|_\lambda \leq 2 \|H(k)\| \|BW_1\| \|e_k\|_\lambda. \tag{32} \]
for some $\lambda$ large enough. Substituting Eq. (31) into Eq. (13), we obtain
\[
e_{k+1}(t) = \left( I_{mN} - (H(k) \otimes CBW_1) \right) e_k(t)
\]
\[
- (I_N \otimes CA) \int_0^t \Delta \chi_k(t) (t - \tau)^{1-\alpha} d\tau - H(k) \otimes CBW_2 \int_0^t e_k(\tau) (t - \tau)^{1-\alpha} d\tau. 
\] (33)

Taking $\lambda$-norm for both side of Eq. (33) via Eq. (32),
\[
\|e_{k+1}\|_{\lambda} \leq \left\| I_{mN} - H(k) \otimes CBW_1 \right\|_{\lambda} + \left\| I_N \otimes CA \right\|_{\lambda} \sqrt{\frac{1}{1 - p(1 - \alpha)}} \frac{1}{\sqrt{p \lambda}} \int_0^t \|H(k)\| \|B\| \|W_1\| \|e_k\|_{\lambda} \lambda - (1 - \alpha) \sqrt{\frac{1}{1 - p(1 - \alpha)}} \frac{1}{\sqrt{p \lambda}} \|e_k\|_{\lambda}.
\]

Finally, for some sufficient large $\lambda$ via Eq. (30) and Eq. (34), we deduce $\lim_{k \to \infty} \|e_k\|_{\lambda} = 0$. The proof is completed. $\square$

**Remark 3.5.** If the iteration-varying are periodic along the iteration axis, then $H(k) = H(i + n\tilde{T}) = H(i)$, where $i = k \mod \tilde{T}$, $n = \left\lfloor \frac{i}{\tilde{T}} \right\rfloor$ and $\tilde{T}$ denotes a period. Obviously, we have
\[
H(k) = \begin{cases}
H(i), & k \mod \tilde{T} = i, \\
H(\tilde{T}), & k \mod \tilde{T} = 0.
\end{cases}
\]

**Remark 3.6.** Theorem 3.4 indicates that the proposed algorithm is still effective encountering the varying graphs, which may be caused by certain environments. The sufficient condition is established on all cases of the varying graphs. That is, the contraction mapping is ensured for arbitrary communication graph. Therefore, the proposed scheme is widely applicable for a general class of FOMAS settings.

### 3.2. Convergency analysis for nonlinear FOMASs

As an extension of the previous subsection, in this subsection, we proceed to consider a $D^\alpha$-type updating rule with forgetting factor for nonlinear FOMASs (3).

We need the following assumptions:

**Assumption 1.** Nonlinear term $f$ satisfies the global Lipschitz condition, this is $\|f(x'(t), t) - f(x''(t), t)\| \leq l_f \|x'(t) - x''(t)\|$ for any $x'$, $x''$ and $t \in [0, T]$.

**Assumption 2.** The following parameters are given: the initial input $u_0(t)$, initial state $x_0(t)$ for every agent, the initial desired input $u_d(t)$, and initial desired state $x_d(t)$ for the virtual leader.

**Assumption 3.** $CB$ is column full rank.

**Lemma 3.7** (see [38]). If $\{a_i\}_{i=0}^\infty$ is a sequence of real number such that
\[
|a_{i+1}| \leq \beta |a_i| + \epsilon, \quad 0 \leq \beta < 1
\]
then
\[
\lim_{i \to \infty} \sup |a_i| \leq \left( \frac{1}{1 - \beta} \right) \epsilon.
\]
Let leader satisfy the following nonlinear model:
\begin{align}
\begin{cases}
\dot{C}_D^q x_d(t) = f(x_d(t), t) + Bu_d(t), \\
y_d(t) = C x_d(t).
\end{cases}
\end{align}
(34)

The distributed measurement $\eta_{k,i}(t)$ is still considered as Section 3.1.

For $i$th agent at the $k$th iteration, we consider the $D^q$-type updating learning rule with forgetting factor:
\begin{equation}
u_{k+1,i}(t) = (1 - \phi_i)u_{k,i}(t) + \phi_iu_{0,i}(t) + W_3 \int_0^t D^q_{t} \eta_{k,i}(t)
\end{equation}
(35)
and initial state updating learning rule:
\begin{equation}x_{k+1,i}(0) = x_{k,i}(0) + W_3 e_{k,i}(0)
\end{equation}
(36)
where $\phi_i \in [0, 1)$ is the forgetting factor, and $u_{0,i}(t)$ is the input of $i$th agent at initial iteration.

Then, the learning rule (35) and (36) are rewritten into the compact form via Kronecker product:
\begin{equation}u_{k+1}(t) = ((I_N - \Phi) \otimes I_m)u_k(t) + (\Phi \otimes I_m)u_0(t) + ((L + S) \otimes W_3) \int_0^t D^q_{t} e_k(t),
\end{equation}
(37)
and
\begin{equation}x_{k+1}(0) = x_k(0) + ((L + S) \otimes W_3)e_k(0),
\end{equation}
(38)
where $\Phi = \text{diag}(\phi_1, \phi_2, \ldots, \phi_N)$, $u_0(t) = \text{diag}(u_{0,1}(t), u_{0,2}(t), \ldots u_{0,N}(t))$.

**Theorem 3.8.** For nonlinear FOMAS (3), considering the update rule (35), (36) and Assumptions 1–3, the consensus tracking error $e_k(t)$ is bounded as $k \to \infty$ provided that
\begin{equation}\| (I_N - \Phi) \otimes I_m - (L + S) \otimes W_3 CB \| < 1,
\end{equation}
(39)
and
\begin{equation}\| I_{Nm} - (L + S) \otimes W_3 C \| < 1.
\end{equation}
(40)
In addition, the state error $\delta x_k(t)$ and input error $\delta u_k(t)$ are bounded.

**Proof.** For Eqs. (3) and (35), the error between $u_d(t)$ coming from Assumption 2 and $u_{k+1,i}(t)$ in compact form at $k + 1$ iteration is
\begin{align}
\delta u_{k+1}(t) &= 1_N \otimes u_d(t) - u_{k+1}(t) \\
&= 1_N \otimes u_d(t) - ((I_N - \Phi) \otimes I_m)u_k(t) - (\Phi \otimes I_m)u_0(t) - ((L + S) \otimes W_3) \int_0^t D^q_{t} e_k(t) \\
& \quad - (\Phi \otimes I_m)(1_N \otimes u_d(t)) + (\Phi \otimes I_m)(1_N \otimes u_d(t)) \\
& = ((I_N - \Phi) \otimes I_m)(1_N \otimes u_d(t) - u_k(t)) + (\Phi \otimes I_m)(1_N \otimes u_d(t) - u_0(t)) \\
& \quad - ((L + S) \otimes W_3)(1_N \otimes C)(1_N \otimes f(x_d(t), t) - \tilde{f}(x_k(t), t)) + (I_N \otimes B)u_d(t) \\
& \quad - (I_N \otimes B)u_k(t) \\
& = ((I_N - \Phi) \otimes I_m - (L + S) \otimes W_3 CB)\delta u_k(t) + (\Phi \otimes I_m)\delta u_0(t) \\
& \quad - ((L + S) \otimes W_3 C)(1_N \otimes f(x_d(t), t) - \tilde{f}(x_k(t), t)),
\end{align}
(41)
where $1_N$ denotes $N$ column vector which elements all are 1 and $\tilde{f}(x_k(t), t) = [f^T(x_{k,1}(t)), f^T(x_{k,2}(t)), \ldots, f^T(x_{k,N}(t))]^T$.

Taking norm for Eq. (41) and using Assumption 1, we have

$$
\|\delta u_{k+1}(t)\| \leq \| (I_N - \Phi) \otimes I_m - (L + S) \otimes W_3 CB \| \| \delta u_k(t) \| + \| \Phi \otimes I_m \| \| \delta u_0(t) \|
+ \| (L + S) \otimes W_3 C \| \| 1_N \otimes f(x_d(t), t) - \tilde{f}(x_k(t), t) \|
\leq \| (I_N - \Phi) \otimes I_m - (L + S) \otimes W_3 CB \| \| \delta u_k(t) \| + \| \Phi \otimes I_m \| \| \delta u_0(t) \|
+ I_f \| (L + S) \otimes W_3 C \| \| \delta x_k(t) \|.
$$  

Taking $\lambda$-norm for Eq. (42), we obtain

$$
\| \delta u_{k+1} \|_\lambda \leq \| (I_N - \Phi) \otimes I_m - (L + S) \otimes W_3 CB \| \| \delta u_k \|_\lambda + \| \Phi \otimes I_m \| \| \delta u_0 \|_\lambda
+ I_f \| (L + S) \otimes W_3 C \| \| \delta x_k \|_\lambda.
$$

Next, $\| \delta x_k \|_\lambda$ need to be estimated.

Considering error between $x_{k,t}(t)$ coming from Eq. (3) and $x_d(t)$ coming from Eq. (34), taking norm and using Assumption 1 and Hölder inequality, we have

$$
\| \delta x_k(t) \| = \| 1_N \otimes x_d(t) - x_k(t) \|
= \| 1_N \otimes (x_d(0) + \frac{1}{\Gamma(\alpha)} \int_0^t f(x_d(\tau), t) + Bu_d(\tau) d\tau)
- (x_k(0) + \frac{1}{\Gamma(\alpha)} \int_0^t \tilde{f}(x_k(\tau), t) + (I_N \otimes B)u_k(\tau) d\tau) \|
= \| \delta x_k(0) + \frac{1}{\Gamma(\alpha)} \int_0^t 1_N \otimes f(x_d(\tau), t) - \tilde{f}(x_k(\tau), t)
\| (t - \tau)^{1-\alpha} d\tau
+ \frac{1}{\Gamma(\alpha)} \int_0^t (I_N \otimes B) \| \delta u_k(\tau) \|
\| (t - \tau)^{1-\alpha} d\tau
\leq \| \delta x_k(0) \| + \frac{I_f}{\Gamma(\alpha)} \int_0^t \| \delta x_k(\tau) \| (t - \tau)^{1-\alpha} d\tau
+ \frac{I_N \otimes B}{\Gamma(\alpha)} \int_0^t \| \delta u_k(\tau) \| (t - \tau)^{1-\alpha} d\tau
\leq \| \delta x_k(0) \| + \frac{I_f}{\Gamma(\alpha)} \frac{1}{1 - p(1 - \alpha)} \| \delta x_k(\tau) \| t^{\frac{1}{\beta}-(1-\alpha)} \| \delta u_k \|_\lambda
+ \frac{I_N \otimes B}{\Gamma(\alpha)} \frac{1}{1 - p(1 - \alpha)} \| \delta u_k \|_\lambda.
$$

Taking $\lambda$-norm, we have

$$
\| \delta x_k \|_\lambda
\leq \| \delta x_k(0) \| + \frac{I_f T^{\frac{1}{\beta}-(1-\alpha)}}{\Gamma(\alpha) \sqrt{q \lambda}} \sqrt{\frac{1}{1 - p(1 - \alpha)}} \| \delta x_k \|_\lambda
+ \frac{I_N \otimes B T^{\frac{1}{\beta}-(1-\alpha)}}{\Gamma(\alpha) \sqrt{q \lambda}} \sqrt{\frac{1}{1 - p(1 - \alpha)}} \| \delta u_k \|_\lambda
= \| \delta x_k(0) \| + I_f \mu_\lambda \| \delta x_k \|_\lambda + \mu_\lambda \| I_N \otimes B \| \| \delta u_k \|_\lambda,
$$  

where $\mu_\lambda = \frac{T^{\frac{1}{\beta}-(1-\alpha)}}{\Gamma(\alpha) \sqrt{q \lambda}} \sqrt{\frac{1}{1 - p(1 - \alpha)}}$. 
So, taking $\lambda > \frac{1}{q} \left( \frac{I_fT^{1/(1-q)}p^{(1-q)}}{\Gamma(\alpha) - \sqrt{\frac{1}{p(1-p)}}} \right)^q$, we have

$$\|\delta x_k\|_\lambda \leq \frac{1}{1 - I_f\mu_\lambda} \|\delta x_k(0)\| + \frac{\mu_\lambda \|I_N \otimes B\|}{1 - I_f\mu_\lambda} \|\delta u_k\|_\lambda.$$ 

Then

$$\|\delta u_{k+1}\|_\lambda \leq \beta \|\delta u_k\|_\lambda + \epsilon,$$

where $\beta = \left( \|I_N - \Phi\| \otimes I_m - (L + S) \otimes W_3CB \right) + \frac{I_f\mu_\lambda \|I_m \otimes (L+S) \otimes W_3CB\|}{1 - I_f\mu_\lambda}$ and $\epsilon = \|\Phi \otimes I_m\| \|\delta u_0\|_\lambda$.

Note Eq. (39) and taking $\lambda > \frac{1}{q} \left( \frac{I_fT^{1/(1-q)}p^{(1-q)}}{\Gamma(\alpha) - \sqrt{\frac{1}{p(1-p)}}} \right)^q$. we have $0 < \hat{\beta} < 1$.

By Eq. (36) and Assumption 3, we have

$$\|\delta x_k(0)\| \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \quad (44)$$

Obviously, $u_{k,t}(t)$ converges o the neighborhood of $u_{d}(t)$ of radius in regard to the $\lambda$-norm. Thus, by Lemma 3.7 and letting $\epsilon \rightarrow \|\Phi \otimes I_m\| \|\delta u_0\|_\lambda$, we have

$$\lim_{k \rightarrow \infty} \sup \|\delta u_k\|_\lambda \leq \frac{\|\Phi \otimes I_m\| \|\delta u_0\|_\lambda}{1 - \hat{\beta}}. \quad (45)$$

By Eqs. (43)–(45), we obtain

$$\lim_{k \rightarrow \infty} \|\delta x_k\|_\lambda \leq \frac{\mu_\lambda \|\Phi \otimes I_m\| \|\delta u_0\|_\lambda}{(1 - I_f\mu_\lambda)(1 - \hat{\beta})}.$$ 

Thus, the tracking error at $k$th iteration is

$$\|e_k\|_\lambda = \|I_N \otimes y_{d}(t) - y_k(t)\|_\lambda$$

$$= \|I_N \otimes C\| \|I_N \otimes x_{d}(t) - x_k(t)\|_\lambda$$

$$\leq \|I_N \otimes C\| \|\delta x_k\|_\lambda$$

$$\leq \|I_N \otimes C\| \left( \frac{1}{1 - I_f\mu_\lambda} \|\delta x_k(0)\| + \frac{\mu_\lambda \epsilon}{(1 - I_f\mu_\lambda)(1 - \hat{\beta})} \right).$$

So,

$$\lim_{k \rightarrow \infty} \sup \|e_k\|_\lambda \leq \frac{\mu_\lambda \|\Phi \otimes I_m\| \|I_N \otimes C\| \|\delta u_0\|_\lambda}{(1 - I_f\mu_\lambda)(1 - \hat{\beta})}.$$
fig. 1. Directed fixed communication topology among agents in the network.

fig. 2. Directed iteration-varying communication topology among agents in the network.

clearly, the supremum of tracking error is related to initial input error and forgetting factor. furthermore, if \( \| \delta u_0 \| \lambda = 0 \) or no forgetting factor is employed, the algorithm converges to the desired trajectories. □

remark 3.9. the result in section 3.2 can be generalized to iteration-varying communicating graph from the fixed graph following similar steps of theorem 3.8. for saving space, the results and proofs are omitted for this generalization.

remark 3.10. theorem 3.8 reveals that the forgetting factor can improve the system robustness. although the final tracking error does not converge to zero but is limited within a small bound around zero; however, we note that the convergence condition is not stronger than the linear case. the inherent reason is that the nonlinearity satisfies the globally lipschitz continuous condition.

4. simulation examples

in this section, we consider a network of five agents to expound the efficacy of the proposed consensus scheme. we set \( \alpha = 0.85 \) and the norm of the tracking errors in each iteration is designated 2-norm in the following examples. the communication graphs for fixed and iteration-varying cases are illustrated in figs. 1 and 2, where vertex 0 represents the virtual leader. it has directed edges to agents 1 and 3. we adopt 0–1 weighting.
In the following examples, the initial state at first iteration is chosen as \( x_{0,1}(0) = [0, -1]^T \), \( x_{0,2}(0) = [1, -2]^T \), \( x_{0,3}(0) = [-2, 0]^T \) and \( x_{0,4}(0) = [-1, 1]^T \). The desired initial state is unique \( x_d(0) = 0 \). The initial control signal \( u_{0,j}(t) = 0 \), \( j = 1, 2, 3, 4 \) for all agents.

**Example 4.1.** Consider the \( i \)th agent model as follows:

\[
\begin{align*}
\xi D_t^2 x_i(t) &= \begin{bmatrix} 0.7 & 0 \\ 0 & 0.9 \end{bmatrix} x_i(t) + \begin{bmatrix} 0.8 & -0.7 \\ 0 & 1 \end{bmatrix} u_i(t), \\
y_i(t) &= \begin{bmatrix} 1.5 \\ 0 \\ 0.9 \end{bmatrix} x_i(t)
\end{align*}
\]

(46)

and the desired reference trajectory \( y_d(t) = \begin{bmatrix} \sin(2\pi t) \\ \sin(2\pi t) \end{bmatrix}, \forall \ t \in [0, 1] \).

For Fig. 1, the Laplacian for followers is \( L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix} \) and \( S = \text{diag}(1, 0, 1, 0) \).

To verify the contraction conditions in Theorem 3.1, we select the learning gain matrix

\( W_1 = \begin{bmatrix} 0.2252 \\ 0.2252 \end{bmatrix} \) and \( W_2 = \begin{bmatrix} 0.1 \\ 0 \\ 0.1 \end{bmatrix} \).

Clearly, \( \|I - (L + S) \otimes CBW_1\| = 0.9162 < 1 \). Thus, the conditions of Theorem 3.1 are valid for Example 4.1.

In Example 4.1, Fig. 3 shows initial state learning of all agents as the iteration number increases. It is seen that the initial states converge to the desired initial state asymptotically around 20th iteration. Fig. 4 shows output profiles of all agents at the 50th interaction. As the iteration number increases, all output profiles converge to the desired trajectory. Fig. 5 depicts the tracking errors along the iteration axis.

**Example 4.2.** We still consider Eq. (46) with the identical desired reference trajectory. Nevertheless, we change fixed communication graph (Fig. 1) into periodic iteration-varying communication graph (Fig. 2), whose period is 4 from Fig. 2.

Clearly,
Fig. 4. Output trajectory of all agents in Example 4.1.

Fig. 5. The 2-norm of the tracking errors for all agents in each interaction in Example 4.1.

\[
H(1) = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & -1 & 1
\end{bmatrix}, \quad H(2) = \begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 2 & -1 \\
0 & 0 & -1 & 1
\end{bmatrix},
\]

\[
H(3) = \begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
0 & 0 & 2 & -1 \\
0 & -1 & -1 & 2
\end{bmatrix}, \quad \text{and} \quad H(4) = \begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & 0 & -1 \\
0 & 0 & 2 & -1 \\
0 & -1 & -1 & 2
\end{bmatrix}.
\]

To verify the contraction conditions in Theorem 3.4, we select the learning gain matrixes similarly to Example 4.1. Clearly, \(\|I - H(1) \otimes CBW_1\| = 0.8983 < 1\), \(\|I - H(2) \otimes CBW_2\| = 0.9162 < 1\), \(\|I - H(3) \otimes CBW_3\| = 0.9564 < 1\) and \(\|I - H(4) \otimes CBW_4\| = 0.9162 < 1\). Thus, the conditions in Theorem 3.4 are valid for Example 4.2.

In Example 4.2, Fig. 6 shows the initial state learning process, where all initial states converge to the desired initial state around 30th iteration. Fig. 7 shows the output profiles of all agents at the 50th iteration. As the iteration number increases, all output profiles converge to the desired trajectory. Fig. 8 depicts the tracking errors in each iteration, which demonstrates the precise tracking performance during a finite time interval.
Fig. 6. Initial state profile vs. iteration number in Example 4.2.

Fig. 7. Output trajectory of all agents in Example 4.2.

Table 1
Comparison of tracking error between Examples 4.1 and 4.2 in 50th iteration.

<table>
<thead>
<tr>
<th>Agent</th>
<th>Example 4.1</th>
<th>Example 4.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0017</td>
<td>0.0011</td>
</tr>
<tr>
<td></td>
<td>0.0014</td>
<td>0.0008</td>
</tr>
<tr>
<td>2</td>
<td>0.0014</td>
<td>0.0010</td>
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<tr>
<td></td>
<td>0.0027</td>
<td>0.0020</td>
</tr>
<tr>
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<td>0.0026</td>
</tr>
<tr>
<td></td>
<td>0.0039</td>
<td>0.0026</td>
</tr>
<tr>
<td>4</td>
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<td>0.0028</td>
</tr>
<tr>
<td></td>
<td>0.0043</td>
<td>0.0032</td>
</tr>
</tbody>
</table>

Remark 4.3. In Table 1, we further compare the tracking error in fixed and iteration-varying communication graphs. At the 50th iteration, the error value in the Example 4.1 greater than 0.005; however, the error value in the Example 4.2 less than 0.004.
Example 4.4. Considering the model of the $i$th agent at the $k$th iteration as follows

\[
\begin{aligned}
\epsilon D_{x}^{\mu} x_{k,i}(t) &= \begin{bmatrix} 0.5 \sin(x_{k,i,1}(t) - 0.8x_{k,i,2}(t)) \\ 0.3x_{k,i,1}(t) + 0.2 \sin(x_{k,i,2}(t)) \end{bmatrix} + \begin{bmatrix} 0.4 & 0.6 \\ 0 & 0.7 \end{bmatrix} u_{i}(t), \quad \forall t \in [0, T] \\
y_{i}(t) &= \begin{bmatrix} 0.8 \\ 0.5 \end{bmatrix} x_{i}(t)
\end{aligned}
\]

The desired reference trajectory $y_{d}(t)$ is also governed by Eq. (47). The desired input $u_{d}(t) = [\sin(\pi t) + 0.1; \sin(\pi t) + 0.1]$; $x_{d}(0) = 0$; $u_{0,i}(t) = 0$; and the forgetting factor is $\Phi = diag(0.1, 0.2, 0.3, 0.4)$.

Now, we need to verify the contraction conditions in Theorem 3.8. Taking $\|x_{k,i}(t)\|_{2} = \sqrt{|x_{k,i,1}(t)|^{2} + |x_{k,i,2}(t)|^{2}}$. Note $f(x_{k,i}(t)) = \begin{bmatrix} 0.5 \sin(x_{k,i,1}(t) - 0.8x_{k,i,2}(t)) \\ 0.3x_{k,i,1}(t) + 0.2 \sin(x_{k,i,2}(t)) \end{bmatrix}$, then

\[
|f_{1}(x'_{k,i}(t)) - f_{1}(x''_{k,i}(t))| = |0.5(\sin(x'_{k,i,1}(t)) - \sin(x''_{k,i,1}(t))) + 0.8(x'_{k,i,2}(t) - x''_{k,i,2}(t))| \\
\leq 0.5|\sin(x'_{k,i,1}(t)) - \sin(x''_{k,i,1}(t))| + 0.8|x'_{k,i,2}(t) - x''_{k,i,2}(t)| \\
\leq 0.5|x'_{k,i,1}(t) - x''_{k,i,1}(t)| + 0.8|x'_{k,i,2}(t) - x''_{k,i,2}(t)|
\]

similarly, we have $|f_{2}(x'_{k,i}(t)) - f_{2}(x''_{k,i}(t))| \leq 0.3|x'_{k,i,1}(t) - x''_{k,i,1}(t)| + 0.2|x'_{k,i,2}(t) - x''_{k,i,2}(t)|$.

So

\[
\|f(x'_{k,i}(t)) - f(x''_{k,i}(t))\|_{2} = \sqrt{|f_{1}(x'_{k,i}(t)) - f_{1}(x''_{k,i}(t))|^{2} + |f_{2}(x'_{k,i}(t)) - f_{2}(x''_{k,i}(t))|^{2}} \\
\leq \sqrt{0.8|x'_{k,i,1}(t) - x''_{k,i,1}(t)|^{2} + 1.14|x'_{k,i,2}(t) - x''_{k,i,2}(t)|^{2}} \\
\leq 1.1\|x'_{k,i}(t) - x''_{k,i}(t)\|_{2}.
\]

We can obtain $\|f(x'_{k,i}(t)) - f(x''_{k,i}(t))\|_{2} \leq 1.1\|x'_{k,i}(t) - x''_{k,i}(t)\|_{2}$ satisfying Assumption 1.

The parameters containing initial input $u_{0,i}(t)$, initial state $x_{0,i}(0)$, initial desired input $u_{d}(t)$ and initial desired state $x_{d}(0)$ are given satisfying Assumption 2.

Note that $CB = \begin{bmatrix} 0.8 & 0.5 \\ 0 & 0.2 \end{bmatrix}\begin{bmatrix} 0.4 & 0.6 \\ 0 & 0.7 \end{bmatrix} = \begin{bmatrix} 0.32 & 0.32 \\ 0 & 0.14 \end{bmatrix}$, then $\text{Rank}(CB) = 2$ satisfies Assumption 3.
Fig. 9. Initial state profile vs number of iteration in Example 4.4.

Fig. 10. Output trajectory of all agents with forgetting factor in Example 4.4.

Fig. 11. The 2-norm of trajectory error of all agents with forgetting factor in Example 4.4.
The learning gain matrix is selected as \( W_3 = \begin{bmatrix} 0.3125 & -0.7813 \\ 0 & 1.2500 \end{bmatrix} \).

Clearly, when the forgetting factor exists, we have \( \| (I_4 - \Phi) \otimes I_2 - (L + S) \otimes W_3CB \| = 0.8288 < 1; \| I_8 - (L + S) \otimes W_3C \| = 0.9045 < 1 \). Thus, the conditions in Theorem 3.8 are valid for Example 4.4.

In Example 4.4, Fig. 9 shows the initial state learning process, where all initial states converge to the desired initial state around 20th iteration. Fig. 10 shows output profiles with forgetting factor of all agents at 100th iteration. From Fig. 10, the trajectory of followers cannot track the trajectory of leader precisely as the number of iteration increases since the forgetting factor exists.

Figs. 11–13 show the 2-norm of tracking errors, input errors, and state errors, respectively, with a forgetting factor in each iteration. Obviously, due to the existence of forgetting factor, the tracking error \( e_k(t) \), the state error \( \delta x_k(t) \) and input error \( \delta u_k(t) \) are bounded around zeros.

Clearly, if \( \Phi = diag([0, 0, 0, 0]) \), i.e., the forgetting factor does not exist, \( \| (I_4 - 0) \otimes I_2 - (L + S) \otimes W_3CB \| = 0.9799 < 1 \) satisfies the condition of Theorem 3.8. The initial state profile tends to desired state profile as the increase of iteration number from Fig. 14. The
Fig. 14. Initial state profile vs. number of iteration without forgetting factor in Example 4.4.

Fig. 15. Output trajectory of all agents without forgetting factor in Example 4.4.

Fig. 16. The 2-norm of trajectory error of all agents without forgetting factor in Example 4.4.
trajectory of followers easily tracks the trajectory of leader as the number of iteration from Fig. 15, and the 2-norm of the tracking error $e_k(t)$, the state error $\delta x_k(t)$, and input error $\delta u_k(t)$ tends to zero as shown in Figs. 16–18.

5. Conclusions

The iterative learning consensus problem has been addressed in this paper for both linear and nonlinear FOMASs. A $PD^\alpha$-type distributed learning scheme is proposed with initial state learning mechanism. Both fixed and iteration-varying communication graphs are considered for linear FOMAS. The sufficient conditions for guaranteeing the asymptotical consensus are provided and proved. For further research, a $D^\alpha$-type updating rule with forgetting factor is provided for the nonlinear FOMAS. Sufficient conditions guarantee that the tracking error, input error, and state error are bounded. Three examples are used to validate the effectiveness of our results.
References